DISSERTATION

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Oscillation Theorems for Semi-Infinite and Infinite Jacobi Operators

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Kerstin Ammann

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Abstract

The Jacobi difference equation (JDE) plays an important role (not just) in mathematical physics: e.g., it containes the one-dimensional discrete Schrödinger equation as a special case and is intimately related to the theory of orthogonal polynomials as well as to continued fractions.

While classical oscillation theory for Jacobi operators puts the sign-changes of solutions of one single operator at the centre of consideration, we compare the number of sign-changes of solutions of two different Jacobi operators. We show that this difference equals the number of weighted sign-changes of the Wronskian of those solutions.

The key discovery in oscillation theory, which goes back to the work of Sturm, is the fact that for any real z the number of sign-changes of a solution $u(z)$ equals the number of eigenvalues of the operator below z. Our theorem refines this observation for the JDE by showing that the number of weighted sign-changes of the Wronskian equals the difference of the number of eigenvalues of the operators in the corresponding interval. The main advantage of this approach is that our theorem is also applicable in gaps of the essential spectrum above its infimum, where the classical theorem breaks down (since the solutions are oscillatory, but the Wronskian isn't).

This theorem is proven for compact, sign-definite perturbations of the potential of Jacobi operators on the line and on the half-line. For the finite case, we extend earlier work for perturbations of the potential to perturbations of all coefficients.

Moreover, we show that this idea carries over to the leading principal minors of Jacobi matrices, which exhibit the same sign pattern as a solution at 0.

Zusammenfassung

Die Jacobi Differenzengleichung (JDG) spielt (nicht nur) in der mathematischen Physik eine wichtige Rolle: z.B. beinhaltet sie die eindimensionale diskrete Schrödingergleichung als Spezialfall und ist eng mit der Theorie der orthogonalen Polynome sowie den Kettenbrüchen verknüpft.
Während die klassische Oszillationstheorie für Jacobi Operatoren die Vorzeichenwechsel der Lösungen eines einzigen Operators ins Zentrum ihrer Betrachtungen stellt, vergleichen wir die Anzahl der Vorzeichenwechsel von Lösungen zweier verschiedener Jacobi Operatoren. Wir zeigen, dass diese Differenz der Anzahl der gewichteten Vorzeichenwechsel der Wronski Determinante der beiden Lösungen entspricht.
Die zentrale Entdeckung der Oszillationstheorie geht zurück auf Sturm und besagt, dass für jedes reelle z die Anzahl der Vorzeichenwechsel einer Lösung $u(z)$ der Anzahl der Eigenwerte des Operators unterhalb von z entspricht. Unser Theorem entwickelt diese Beobachtung für die JDG dahingehend weiter, dass es zeigt, dass die Anzahl der gewichteten Vorzeichenwechsel der Wronski Determinante der Differenz der Anzahl der Eigenwerte der beiden Operatoren im zugehörigen Intervall entspricht. Der Vorteil dabei ist, dass unser Theorem auch in Lücken des wesentlichen Spektrums überhalb seines Infimums anwendbar ist, im Gegensatz zum klassischen Theorem (da die Lösungen hier oszillatorisch sind, die Wronski Determinante aber nicht).
Dieses Theorem wird für kompakte, vorzeichenbestimmte Störungen des Potentials von singulären Jacobi Operatoren, wie auch von Jacobi Operatoren mit einem regulären Endpunkt, bewiesen. Für den endlichen Fall erweitern wir frühere Arbeiten über Störungen des Potentials auf Störungen aller Koeffizienten. Weiters zeigen wir, dass sich diese Idee auch auf die führenden Hauptminoren von Jacobi Matrizen übertragen lässt, da sie das selbe Vorzeichenmuster aufweisen wie eine Lösung bei 0.

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Contents

Chapter 1

Introduction

In this thesis we present new oscillation theorems for a particular discrete equation, namely the *Jacobi difference equation (JDE)*,

$$\tau u = zu, \tag{1.1}$$

where $z \in \mathbb{R}$,

$$\begin{aligned} \tau : \ell(\mathbb{Z}) &\to \ell(\mathbb{Z}), \\ u(n) \mapsto (\tau u)(n) &= a(n)u(n+1) + a(n-1)u(n-1) + b(n)u(n) \\ &= \partial(a(n-1)\partial u(n-1)) + (b(n) + a(n) + a(n-1))u(n), \end{aligned} \tag{1.2}$$

and where $\ell(I) = \{\varphi \mid \varphi : I \subseteq \mathbb{Z} \to \mathbb{R}\}$ is the space of real-valued sequences and $\partial\varphi(n) = \varphi(n+1) - \varphi(n)$ is the usual forward difference operator.

The JDE can be viewed as the discrete counterpart of the famous Sturm–Liouville differential equation, $\tau u = zu$, where

$$\tau = \frac{1}{r(x)} \left(-\frac{\mathrm{d}}{\mathrm{dx}} p(x) \frac{\mathrm{d}}{\mathrm{dx}} + q(x) \right). \tag{1.3}$$

Setting $a = 1$ (that is $p = r = 1$ in the continuous case) we obtain the one-dimensional Schrödinger equation as a special case. Besides that, Jacobi operators appear at various other occasions in mathematics, physics and engineering: they constitute a simple one-band tight binding model in quantum mechanics [9], a model for a chain of masses coupled via springs and fixed at both end points, or for a rod vibrating in longitudinal motion [20]; they are closely related to orthogonal polynomials on the real line as well as to continued fractions [22, 10] and they play a fundamental role in the investigation of the Toda and the Kac-van Moerbeke lattices [41]. A comprehensive introduction to Jacobi operators can be found in [42] and for a more general treatment of difference equations, as well as discrete oscillation theory, and boundary value problems, we refer for example to [15, 26], [1], and [5], respectively.

A key observation of oscillation theory for Sturm–Liouville operators, as well as for Jacobi operators [18], is the famous oscillation theorem, which goes back to the seminal work of Sturm from 1836 [40] and states that the n-th eigenfunction has exactly $n-1$ sign-changes (*nodes*). But the fact that above the infimum of the essential spectrum of a Sturm–Liouville operator, and also of a Jacobi operator, all solutions are oscillatory (i.e., they have infinitely many nodes) has brought up the question how oscillation theory can be extended to gaps of the essential spectrum above its infimum, since a naïve use of course leads to $\infty - \infty$. This problem has first been overcome by Gesztesy, Simon, and Teschl in [19] where they showed that the number of eigenvalues of a Sturm–Liouville operator in a gap of the essential spectrum equals the number of sign-changes of the Wronskian of two suitable solutions, see also [35, 48] for a review of the continuous case and its discrete counterpart [46].
We will extend this concept to *perturbations* of Jacobi operators in the following sense: we show that the number of weighted nodes of the Wronski determinant (which we will call the *relative nodes*) of two suitable solutions of two different JDEs equals the number of eigenvalues the perturbation inserts into or removes from a gap of the essential spectrum. In the continuous case the link to perturbation theory has been established in [29, 30], which already led to new eigenvalue asymptotics [27] and relative oscillation criteria [28].

Before we go into further details and make rigorous statements, we recall some basic principles on which our considerations rely. The spectral problems arising from the JDE, where we impose either Dirichlet boundary conditions at finite points or square summability near infinite endpoints, are formulated in terms of Jacobi matrices: we consider *infinite* Jacobi operators,

$$\begin{aligned} H : \ell^2(\mathbb{Z}) &\to \ell^2(\mathbb{Z}) \\ \psi &\mapsto \tau\psi, \end{aligned} \tag{1.4}$$

given by the infinite matrix

$$H = \begin{pmatrix} \ddots & \ddots & \ddots & & & \\ & a(n-1) & b(n) & a(n) & & \\ & & a(n) & b(n+1) & a(n+1) & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1.5}$$

semi-infinite Jacobi operators,

$$\begin{aligned} H_\pm : \ell^2(\pm\mathbb{N}) &\to \ell^2(\pm\mathbb{N}) \\ \psi &\mapsto \tau\psi, \end{aligned} \tag{1.6}$$

associated with

$$H_+ = \begin{pmatrix} b(1) & a(1) & \\ a(1) & b(2) & \ddots \\ & \ddots & \ddots \end{pmatrix}, \qquad H_- = \begin{pmatrix} \ddots & \ddots & \\ \ddots & b(-2) & a(-2) \\ & a(-2) & b(-1) \end{pmatrix}, \tag{1.7}$$

and *finite* Jacobi matrices

$$J = \begin{pmatrix} b(1) & a(1) & & \\ a(1) & b(2) & \ddots & \\ & \ddots & \ddots & a(N-2) \\ & & a(N-2) & b(N-1) \end{pmatrix}. \tag{1.8}$$

We assume that $a, b \in \ell^\infty(\mathbb{Z})$ and thus all the mentioned operators are bounded. Moreover, it's well-known that they are self-adjoint (hence the spectrum is contained in the real axis) and that their point spectra are simple, confer e.g. [42]. The spectrum of a Jacobi matrix remains unchanged if we alter signs in the sequence a, but, since the signs of the solutions depend on a from now on we assume $a(n) < 0$ for all n unless we state something else explicitly.
The solution space of the Jacobi difference equation is two-dimensional and by a *solution* $u = u(z)$ of $\tau u = zu$ we will always mean a nontrivial one, i.e., we exclude the case $u = 0$. Hence, a solution u cannot have two consecutive zeros. From now on we denote solutions $u(z)$ fulfilling the right/left boundary condition of the corresponding operator (which will be evident from the context) by $u_\pm(z)$. A short calculation shows that a solution $u(z)$ of $\tau u = zu$, or precisely the projection of $u(z)$ into the corresponding subspace $\ell((0, N))$, is an eigenvector of J if and only if $u(z)$ fulfills $u(z, 0) = u(z, N) = 0$.
Solutions fulfilling $u_\pm(z) \in \ell^2(\pm\mathbb{N})$ are called *Weyl solutions* and exist for all $z \notin \sigma_{ess}(H_\pm)$, where $\sigma_{ess}(H_\pm)$ denotes the essential spectrum of $H_\pm$. Throughout our considerations, the spectral parameter z will always be in a gap of the essential spectrum, hence the solutions $u_\pm(z)$ always exist when we need them (recall that $\sigma_{ess}(H) = \sigma_{ess}(H_-) \cup \sigma_{ess}(H_+)$ holds).

Let $u_j = u_j(z_j)$ be a solution of the JDE $\tau_j u = z_j u$, where $j = 0, 1$. Then we define their (modified) *Wronskian* as the sequence $W(u_0, u_1) \in \ell(\mathbb{Z})$, where

$$W_n(u_0, u_1) = a(n)(u_0(n)u_1(n+1) - u_1(n)u_0(n+1)) \tag{1.9}$$

for all $n \in \mathbb{Z}$. At each point n we weight

$$\#_n(u_0, u_1) = \begin{cases} 1 & \text{if } b_0(n+1) - z_0 - b_1(n+1) + z_1 > 0 \text{ and} \\ & \quad \text{either } W_n(u_0, u_1)W_{n+1}(u_0, u_1) < 0 \\ & \quad \text{or } W_n(u_0, u_1) = 0 \text{ and } W_{n+1}(u_0, u_1) \neq 0 \\ -1 & \text{if } b_0(n+1) - z_0 - b_1(n+1) + z_1 < 0 \text{ and} \\ & \quad \text{either } W_n(u_0, u_1)W_{n+1}(u_0, u_1) < 0 \\ & \quad \text{or } W_n(u_0, u_1) \neq 0 \text{ and } W_{n+1}(u_0, u_1) = 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.10}$$

and say the Wronskian has a *(weighted) node* at n if $\#_n(u_0, u_1) \neq 0$.

The main aim of this thesis is, to prove the *relative oscillation theorem* for infinite Jacobi operators, which is

Theorem 1.1. *Let $a_0 = a_1 < 0$ and let $a_j, b_j \in \ell^\infty(\mathbb{Z})$, where $j = 0, 1$, such that $\lim_{n\to\pm\infty} b_0(n) = b_1(n)$ and $b_0(n) \geqslant b_1(n)$ for all $|n| > N$ and some N. Then, for each $z \notin \sigma_{ess}(H_0)$ the number of weighted nodes of the Wronskian*

$$\mathcal{N}(z) = \sum_{n=-\infty}^{\infty} \#_n(u_{0,+}(z), u_{1,-}(z)) - \begin{cases} 1 & \textit{if } W(u_{0,+}(z), u_{1,-}(z)) \\ & \quad \textit{vanishes near } -\infty \\ 0 & \textit{otherwise} \end{cases} \tag{1.11}$$

$$= \sum_{n=-\infty}^{\infty} \#_n(u_{0,-}(z), u_{1,+}(z)) - \begin{cases} 1 & \textit{if } W(u_{0,-}(z), u_{1,+}(z)) \\ & \quad \textit{vanishes near } -\infty \\ 0 & \textit{otherwise} \end{cases} \tag{1.12}$$

is finite, and if moreover $[z_-, z_+] \cap \sigma_{ess}(H_0) = \emptyset$, then

$$E_{[z_-,z_+)}(H_1) - E_{(z_-,z_+]}(H_0) = \mathcal{N}(z_+) - \mathcal{N}(z_-), \tag{1.13}$$

and if $z < \inf \sigma_{ess}(H_0)$, then

$$E_{(-\infty,z)}(H_1) - E_{(-\infty,z]}(H_0) = \mathcal{N}(z), \tag{1.14}$$

where $E_\Omega(H_j)$ is the number of eigenvalues of H_j in $\Omega \subseteq \mathbb{R}$, and $u_{j,\pm}(z)$ are corresponding Weyl solutions, i.e., $u_{j,\pm}(z) \in \ell^2(\pm\mathbb{N})$.

Thereto, recall that

$$\sigma_{ess}(H_0) = \sigma_{ess}(H_1) \tag{1.15}$$

and also $\sigma_{ess}(H^0_\pm) = \sigma_{ess}(H^1_\pm)$ holds since the perturbation is compact. We moreover assumed that $b_0 - b_1$ is *sign-definite* near infinite endpoints to ensure that the limits exist. In Chapter 9 we present further oscillation theorems for infinite Jacobi operators and $z < \inf \sigma_{ess}(H_0)$.

Hence, we notice that as z increases, each of the Wronskians $W(u_{0,+}(z), u_{1,-}(z))$ and $W(u_{0,-}(z), u_{1,+}(z))$ receives a new node at each eigenvalue of H_1 and loses a node at each eigenvalue of H_0. At each z in both resolvent sets, the number of nodes remains unchanged and for each z in both spectra the Wronskians lose a node locally, that is, $\mathcal{N}(z-\varepsilon) = \mathcal{N}(z) + 1 = \mathcal{N}(z+\varepsilon)$.

Our next objective is, to establish the relative oscillation theorem also for semi-infinite Jacobi operators:

Theorem 1.2. *Let $a_0 = a_1 < 0$ and let $a_j, b_j \in \ell^\infty(\mathbb{N})$, where $j = 0, 1$, such that $\lim_{n\to\infty} b_0(n) = b_1(n)$ and $b_0(n) \geqslant b_1(n)$ for all $n > N$ and some N. Then, for each $z \notin \sigma_{ess}(H^0_+)$ the number of weighted nodes of the Wronskian*

$$\begin{aligned} N(z) &= \sum_{n=0}^{\infty} \#_n(u_{0,+}(z), u_{1,-}(z)) - \begin{cases} 1 & \text{if } W_0(u_{0,+}(z), u_{1,-}(z)) = 0 \\ 0 & \text{otherwise} \end{cases} \\ &= \sum_{n=0}^{\infty} \#_n(u_{0,-}(z), u_{1,+}(z)) - \begin{cases} 1 & \text{if } W_0(u_{0,-}(z), u_{1,+}(z)) = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{1.16}$$

is finite, and if moreover $[z_-, z_+] \cap \sigma_{ess}(H^0_+) = \emptyset$, then

$$E_{[z_-,z_+)}(H^1_+) - E_{(z_-,z_+]}(H^0_+) = N(z_+) - N(z_-), \tag{1.17}$$

and if $z < \inf \sigma_{ess}(H^0_+)$, then

$$E_{(-\infty,z)}(H^1_+) - E_{(-\infty,z]}(H^0_+) = N(z), \tag{1.18}$$

where $E_\Omega(H^j_+)$ is the number of eigenvalues of H^j_+ in $\Omega \subseteq \mathbb{R}$, and $u_{j,\pm}(z)$ are solutions fulfilling the right/left boundary condition of H^j_+, i.e., $u_{j,+} \in \ell^2(\mathbb{N})$ and $u_{j,-}(0) = 0$.

We present further oscillation theorems for semi-infinite Jacobi operators and $z < \inf \sigma_{ess}(H^0_+)$ in Chapter 9.

Now we briefly review the proof of these two theorems. In Chapter 7 we show that the Wronskian has at most finitely many weighted nodes in gaps of the essential spectrum. In doing so, we also study Wronskians of solutions corresponding to two different spectral parameters, which generalizes earlier findings from [46] to the case of two different Jacobi operators.

In particular, the following should be mentioned: if there are at most finitely many eigenvalues in a gap (z_-, z_+) of the essential spectrum of H_0, then the Wronskian $W(u_0(z_-), u_1(z_+))$ is oscillatory if and only if the perturbation inserts an infinite number of eigenvalues into the gap (which of course accumulate at the boundary). Therefore, see the following theorem, which we prove in Chapter 7:

Theorem 1.3. *Let* $a_0 = a_1 < 0$, $\lim_{n\to\pm\infty} b_0(n) = b_1(n)$, *and* $b_0(n) \geqslant b_1(n)$ *for all* $|n| \geqslant N$ *and some* N. *Then, for all* $z_-, z_+ \in \mathbb{R}$, $z_- < z_+$, *such that* $\dim \operatorname{Ran} P_{(z_-,z_+)}(H_0) < \infty$ *holds we have*

$$\sum_{n=-\infty}^{\infty} \#_n(u_0(z_-), u_1(z_+)) < \infty \qquad \Longleftrightarrow \qquad \dim \operatorname{Ran} P_{(z_-,z_+)}(H_1) < \infty, \quad (1.19)$$

where $u_j(z_\pm)$ *are (arbitrary) solutions of* $\tau_j u = z_\pm u$, $j = 0, 1$, *and* $P_\Omega(H_j)$ *denote the spectral projections of* H_j, $\Omega \subseteq \mathbb{R}$. *The same holds for* $H_\pm$ *if we count the nodes at* $\pm\mathbb{N}$.

The next step in the proofs of Theorem 1.1 and Theorem 1.2 is based on the relative oscillation theorem for finite Jacobi matrices from [4], confer Theorem 1.4. We look at (suitably modified) finite Jacobi matrices of sufficiently large dimensions and the (suitably modified) corresponding Wronskians, where the modification is such, that we adapt the right boundary condition of the finite matrix to the Weyl solution u_+. Using the approximation technique which we develop comprehensively in Chapter 8, we then show, that the number of eigenvalues in the considered gap, as well as (the number of nodes of) the Wronskians, converge in some sense to their semi-infinite counterparts. The continuous counterpart of such a technique has already been applied in the Sturm–Liouville [29, 30] and in the Dirac case [37] and goes back to Stolz and Weidmann [38], see also [50]. The present discrete case extends [42, 46]. This already leads to the oscillation theorems for Wronskians, established in Chapter 9, which hold below the essential spectrum.

However, above the infimum of the essential spectrum the situation differs dramatically since we have to approximate two Wronskians at once, but the Weyl solution (which generates the boundary conditions for the finite matrices) corresponds only to one of them. And hence, we don't obtain enough information on the second one as well as on the corresponding endpoint of the interval, but due to the sign-definiteness of the perturbation, we obtain at least an inequality. Approximating twice (at both endpoints of the interval) means that we end up with two inequalities which aren't sharp enough to obtain the theorem. A closer look at the approximation shows, that a possible eigenvalue at a foreign endpoint of the half-open interval actually is approximated from the 'wrong' side, i.e., a possible eigenvalue at the closed endpoint is approximated from outside the spectral interval under consideration such that it doesn't appear in the finite spectra but suddenly in the limit spectrum. Thus, for semi-infinite Jacobi operators we obtain a first version of Theorem 1.2 in Section 10.1, but with the additional assumption

$$z_- \notin \sigma(H_+^1) \quad \text{and} \quad z_+ \notin \sigma(H_+^0). \tag{1.20}$$

To get rid of (1.20), we develop a new strategy in Section 10.2 which (until now) has no Sturm–Liouville or Dirac counterpart: a symmetry argument shows that it's enough to look at the vicinity of a point which is in both spectra. In doing so, we perturb one of the semi-infinite operators 'slightly' near the regular endpoint to move the eigenvalue away from the original position such that we can apply the theorem we already have. Since this perturbation is limited to one of the operators and to the vicinity of the regular endpoint, both Wronskians change only locally, namely at the position 0, where we can explicitly compute that the Wronskian at the original eigenvalue wins a node. This completes the proof of Theorem 1.2.

In Section 11.1 we approximate infinite Jacobi operators by semi-infinite Jacobi operators and obtain Theorem 1.1 with the additional assumption

$$z_- \notin \sigma(H_1) \quad \text{and} \quad z_+ \notin \sigma(H_0) \tag{1.21}$$

in a similar manner. And again we consider the case of a common eigenvalue at the boundary of the spectral interval. Now we have to refine our perturbation argument a bit: since there's no regular endpoint, the Wronskian now changes at infinitely many points as soon as we perturb the operator, which cannot be computed explicitly.
But if we perturb the operator sufficiently far on the left, we can ensure that the Wronskian at the original eigenvalue cannot lose nodes since the perturbation is sign-definite. And since the eigenvalue is approximated, we can perturb one of the operators 'slightly' at a sufficiently small point in $\mathbb{Z}$ where the Weyl solution (taken at a suitable point on the real axis which is moreover sufficiently near to the original eigenvalue) of the second Wronskian vanishes and hence the second Wronskian remains unchainged. Thus, this provides exactly the missing inequality to eliminate (1.21) and hence to prove our main theorem.

Now, we moreover want to introduce our extensions of the relative oscillation theorem for finite Jacobi matrices from [3, 4]. Therefore, recall the following

Theorem 1.4. *Confer Theorem 1.2 in [4]. Let* $a_0 = a_1 < 0$*, then*

$$\begin{aligned} &E_{(-\infty,z_1)}(J_1) - E_{(-\infty,z_0]}(J_0) \\ &\quad = \sum_{j=0}^{N-1} \#_j(u_{0,+}(z_0), u_{1,-}(z_1)) - \begin{cases} 1 & \textit{if } W_0(u_{0,+}(z_0), u_{1,-}(z_1)) = 0 \\ 0 & \textit{otherwise} \end{cases} && (1.22) \\ &\quad = \sum_{j=0}^{N-1} \#_j(u_{0,-}(z_0), u_{1,+}(z_1)) - \begin{cases} 1 & \textit{if } W_0(u_{0,-}(z_0), u_{1,+}(z_1)) = 0 \\ 0 & \textit{otherwise} \end{cases} && (1.23) \end{aligned}$$

holds, where $E_\Omega(J_j), j = 0, 1,$ *is the number of eigenvalues of* J_j *in* $\Omega \subseteq \mathbb{R}$*, and* $u_{j,\pm}(z_j)$ *are solutions fulfilling the right/left Dirichlet boundary condition of* J_j*, i.e.,* $u_{j,+}(z_j, N) = u_{j,-}(z_j, 0) = 0$*.*

First of all, we allow different a's. Therefore, we extend the definition of the Wronskian to

$$W_n(u_0, u_1) = u_0(n)a_1(n)u_1(n+1) - u_1(n)a_0(n)u_0(n+1) \tag{1.24}$$

and the weighting of the relative nodes to

$$\#_n(u_0, u_1) = \begin{cases} 1 & \text{if } W_{n+1}(u_0, u_1)u_0(n+1)u_1(n+1) > 0 \text{ and} \\ & \quad \text{either } W_n(u_0, u_1)W_{n+1}(u_0, u_1) < 0 \\ & \quad \text{or } W_n(u_0, u_1) = 0 \text{ and } W_{n+1}(u_0, u_1) \neq 0 \\ -1 & \text{if } W_n(u_0, u_1)u_0(n+1)u_1(n+1) > 0 \text{ and} \\ & \quad \text{either } W_n(u_0, u_1)W_{n+1}(u_0, u_1) < 0 \\ & \quad \text{or } W_n(u_0, u_1) \neq 0 \text{ and } W_{n+1}(u_0, u_1) = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{1.25}$$

Of course, if $a_0 = a_1$, then the Wronskian as well as the counting method reduce to those introduced in [4] which are (1.9) and (1.10). And since we not just extend the theorem to different a's, but also to more general spectral intervals we define the *number of relative nodes between* m *and* n as

$$\#_{[m,n]}(u_0, u_1) = \sum_{j=m}^{n-1} \#_j(u_0, u_1) \tag{1.26}$$

for all $m < n$. If there are no zeros of the Wronskian at the endpoints m and n, then we have $\#_{[m,n]}(u_0, u_1) = -\#_{[m,n]}(u_1, u_0)$, but otherwise we have to distinguish the following cases: we set

$$\#_{(m,n]}(u_0,u_1) = \#_{[m,n]}(u_0,u_1) - \begin{cases} 1 & \text{if } W_m(u_0,u_1)=0 \\ 0 & \text{otherwise,} \end{cases} \tag{1.27}$$

$$\#_{[m,n)}(u_0,u_1) = \#_{[m,n]}(u_0,u_1) + \begin{cases} 1 & \text{if } W_n(u_0,u_1)=0 \\ 0 & \text{otherwise,} \end{cases} \tag{1.28}$$

and

$$\begin{aligned} \#_{(m,n)}(u_0,u_1) = \#_{[m,n]}(u_0,u_1) &- \begin{cases} 1 & \text{if } W_m(u_0,u_1)=0 \\ 0 & \text{otherwise} \end{cases} \\ &+ \begin{cases} 1 & \text{if } W_n(u_0,u_1)=0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{1.29}$$

Note that we slightly changed the notation compared to [4]: $\#_{(m,n)}$ from [4] is now denoted as $\#_{(m,n]}$.

With these definitions in mind, we find the desired theorem which will appear in [2]:

Theorem 1.5. *Let $a_0, a_1 < 0$, then*

$$\begin{aligned} &E_{(-\infty,z_1)}(J_1) - E_{(-\infty,z_0]}(J_0) \\ &\quad = \#_{(0,N-1]}(u_{0,+}(z_0),u_{1,-}(z_1)) = \#_{(0,N-1]}(u_{0,-}(z_0),u_{1,+}(z_1)) \end{aligned} \tag{1.30}$$

and

$$\begin{aligned} &E_{(-\infty,z_1)}(J_1) - E_{(-\infty,z_0)}(J_0) \\ &\quad = \#_{[0,N-1]}(u_{0,+}(z_0),u_{1,-}(z_1)) = \#_{(0,N-1)}(u_{0,-}(z_0),u_{1,+}(z_1)), \\ &E_{(-\infty,z_1]}(J_1) - E_{(-\infty,z_0]}(J_0) \\ &\quad = \#_{(0,N-1)}(u_{0,+}(z_0),u_{1,-}(z_1)) = \#_{[0,N-1]}(u_{0,-}(z_0),u_{1,+}(z_1)), \\ &E_{(-\infty,z_1]}(J_1) - E_{(-\infty,z_0)}(J_0) \\ &\quad = \#_{[0,N-1)}(u_{0,+}(z_0),u_{1,-}(z_1)) = \#_{[0,N-1)}(u_{0,-}(z_0),u_{1,+}(z_1)) \end{aligned} \tag{1.31}$$

holds for the Wronskian (1.24) with the weighting (1.25) if we set the additional value $a_0(N-1) = a_1(N-1) < 0$ to compute $u_{j,-}(z_j,N)$, where $j = 0,1$.
The number of eigenvalues of J_j in $\Omega \subseteq \mathbb{R}$ is $E_\Omega(J_j)$, and $u_{j,\pm}(z_j)$ are solutions fulfilling the right/left Dirichlet boundary condition of J_j, that is $u_{j,+}(z_j,N) = u_{j,-}(z_j,0) = 0$.

Theorem 1.5 also sharpens Theorem 1.4 where we've counted one weight too

much, namely $\#_{N-1}$. Only therefore we've set $a_0(N-1) = a_1(N-1)$, which obviously doesn't influence J and $\sigma(J)$, but the value $u_{j,-}(z_j, N)$, $j = 0, 1$, depends on it. However, if we drop this assumption, then we have to take the weight at $N-1$ into account. We note that case in Theorem 4.6. On the other hand, for a computation of $u_{j,+}(z_j, 0)$ any negative values $a_0(0)$ and $a_1(0)$ will do the job.
The proof of this theorem is based on the discrete Prüfer transformation where now the difference of the Prüfer angle is put at the center of considerations since it counts the relative nodes. This technique is presented in the chapters 3 and 4 and extends the one from [4].

Compared to [3, 4, 29, 30, 37], we present a simplified proof which eliminates the need to interpolate between operators. This is of particular importance in the present case, since $a_0 < a_1$ doesn't imply the corresponding relation for the operators. For this, simply look at the eigenvalues $-\varepsilon$ and ε of the Jacobi matrix

$$\begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix} \tag{1.32}$$

which move in different directions as ε increases. Hence, the interpolation step would be more difficult since we cannot assume that the Prüfer angle is non-decreasing which is the key ingredient of the mentioned proofs. We refer to the appendix for a computation of the derivative of the Prüfer angle of a linear interpolation of Jacobi matrices (for different Prüfer transformations). This demonstrates that the (suitably transformed) Prüfer angle is strictly increasing if the perturbation matrix is positive definite and extends the corresponding formulas from [4, 46] to different a's and b's.
The proofs for regular Sturm–Liouville operators [29, Theorem 2.3] and regular Dirac operators [37, Theorem 3.3] can be shortened in the same manner and both theorems can be extended to (half-)open and closed spectral intervals analogously to (1.31), which is new. An adapted version of the Sturm–Liouville case can already be found in the recent book [45].

For an extension of Sturm's comparison theorem to relative nodes, we refer to Chapter 6 and [2] (the case $a_0 = a_1$ can be found in [3]). In contrast to the Sturm–Liouville case [29], we don't obtain a direct dependance on the coefficients of the operators as soon as we look at different a's, but the theorem holds if we assume $J_0 \geqslant J_1$ instead.

Finally, from a linear algebra point of view we want to add the following (confer therefore Chapter 5):

Sturm's oscillation theorem also has a determinantal counterpart for hermitian matrices with nonzero (up to the rank of the matrix) leading principle minors: it was found in C. G. J. Jacobi's handwritten legacy (in terms of quadratic forms) and posthumously communicated by Borchardt in 1857 [8]. Later, it has been extended by Gundelfinger in 1881 [23] and Frobenius in 1894 [16], allowing simple and two consecutive zeros in the sequence of leading principle minors, respectively. A direct extension to three or more consecutive zeros isn't possible, therefore confer e.g. [31], where these theorems can also be found in terms of determinants.

Applying the Jacobi-Gundelfinger theorem to Jacobi matrices, we easily obtain Sturm's oscillation theorem with the help of a formula which connects the solutions of the JDE to the leading principle minors of the Jacobi matrix. This moreover proves rigorously that the assumption $a < 0$ can be weakened to $a \neq 0$ if the definition of a node is slightly modified. Such a modification of the definition of a node has already been suggested in [46].
Gantmacher and Krein's proof of Sturm's oscillation theorem for $a < 0$ used the concept of Sturm chains to obtain the determinantal counterpart, confer Theorem II.1.7° in [18]; and in [52, 5.38] it has been deduced from the strict separation of the eigenvalues, but I didn't find a proof in the literature which is based on Jacobi's theorem (although Jacobi's theorem applies to a larger class of matrices).
It remains to remark that it seems to be more natural to look at the leading principal minors of $J - z$ instead of the solutions, since there the nodes can be defined independently of the (sign of the) matrix elements.

As a special case thereof (hence going back to Gantmacher-Krein [18] and Jacobi [8]) in my view the following should also be pointed out: in the Jacobi case, Sylvester's criteria for positive and negative definite symmetric matrices extend to semi-definite matrices (which is well-known not to hold generally for hermitian matrices). I didn't find this note in the literature, although usually

$$\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}, \quad x < 0, \tag{1.33}$$

which is a tridiagonal matrix, is stated as a counterexample for the general case, e.g. in [6, 7, 17, 18, 32]. Hence, in Section 5.4 a short, self-contained proof is presented which shows how this claim extends to the leading principal minors of submatrices of arbitrary tridiagonal matrices.

It remains to mention that Theorem 1.5 of course also carries over to leading principle minors of $J - z$ and we state a rigorous theorem for the case $a_0 = a_1$ in Chapter 5.

As a concluding remark we want to mention that relative oscillation theory has already been extended to Dirac operators in [37, 47] and to symplectic eigenvalue problems in [11, 12, 13, 14] and several other extensions are thinkable, e.g. to CMV matrices. Only recently, Šimon Hilscher pointed out in [36] that an extension to the case of Jacobi difference equations with a nonlinear dependance on the spectral parameter would be of particular interest. Extensions to nodal domains on graphs are currently in preparation and we hope that this work will stimulate further research, e.g. to find new relative oscillation criteria and eigenvalue asymptotics as in the Sturm–Liouville case [27, 28].

Chapter 2

Preliminaries

In this chapter we recall some basic knowledge which we will frequently use in the sequel, in particular the notions of spectra, resolvents, and operator convergence for self-adjoint linear operators in Hilbert spaces will be introduced. For a more comprehensive treatment we refer e.g. to [25, 33, 43, 49, 51] where the herein recalled concents can also be found.
We further introduce Jacobi operators and have a closer look at their Green functions, Weyl solutions and Weyl m-functions, therefore confer e.g. the monograph [42].

2.1 Linear operators

Since the Jacobi matrices considered here are bounded self-adjoint operators in ℓ^2 we will mainly focus on the case of bounded operators in a separable Hilbert space $\mathscr{H}$. Nevertheless, we introduce the basic concepts also for unbounded operators, since, as we will see, many of the intermediate results can be obtained for the unbounded case with almost no additional effort.

Definition 2.1. *A* linear operator *A is a linear mapping* $A : \mathscr{D}(A) \to \mathscr{H}$ *where the* domain *of A,* $\mathscr{D}(A)$*, is a linear subspace of* $\mathscr{H}$*. If the* (operator) norm *of A,*

$$\|A\| = \sup_{\varphi \mid \|\varphi\|=1} \|A\varphi\|, \tag{2.1}$$

is finite, then A is called bounded.

The set

$$\mathscr{L}(\mathscr{H}) = \{A : \mathscr{H} \to \mathscr{H} \mid \sup_{\varphi \mid \|\varphi\|=1} \|A\varphi\| < \infty\} \tag{2.2}$$

is a Banach space. If $\overline{\mathscr{D}(A)} = \mathscr{H}$, then a bounded linear operator A can be uniquely extended to a bounded linear operator $\overline{A} : \mathscr{H} \to \mathscr{H}$ with the same bound by the B.L.T. theorem (Theorem I.7 in [33]).

Definition 2.2. *Let* $\overline{\mathscr{D}(A)} = \mathscr{H}$. *The* adjoint operator A^* *is given by*

$$\mathscr{D}(A^*) = \{\psi \in \mathscr{H} \mid \forall\ \varphi \in \mathscr{D}(A):\ \exists\ \tilde{\psi} \in \mathscr{H} : \langle \psi, A\varphi\rangle = \langle \tilde{\psi}, \varphi\rangle\} \tag{2.3}$$
$$A^*\psi = \tilde{\psi}.$$

An operator A *is called* self-adjoint *if* $A = A^*$.

Lemma 2.3. *Confer Theorem VI.3 in [33]. We have*

- $A \mapsto A^*$ *is a conjugate linear isometric isomorphism of* $\mathscr{L}(\mathscr{H})$ *onto* $\mathscr{L}(\mathscr{H})$.
- $(AB)^* = B^*A^*$
- $(A^*)^* = A$
- *If* $A^{-1} \in \mathscr{L}(\mathscr{H})$, *then* $(A^*)^{-1} \in \mathscr{L}(\mathscr{H})$ *and* $(A^*)^{-1} = (A^{-1})^*$.

Definition 2.4. *The dimension of the range of* A *is called the* rank *of* A, *that is*

$$\operatorname{rank}(A) = \dim \operatorname{Ran}(A). \tag{2.4}$$

An operator $A \in \mathscr{L}(\mathscr{H})$ *is called a* finite rank operator *if* $\dim \operatorname{Ran}(A) < \infty$.

The range and the kernel of A are subspaces of $\mathscr{H}$ and the kernel of A^* is the orthogonal complement of the range of A, that is

$$\operatorname{Ran}(A)^{\perp} = \operatorname{Ker}(A^*). \tag{2.5}$$

Hence, $\operatorname{Ker}(A)$ is closed, whereas $\operatorname{Ran}(A)$ isn't necessarily closed.

Definition 2.5. *The set of* compact operators *is given by*

$$\mathscr{C}(\mathscr{H}) = \overline{\{A \in \mathscr{L}(\mathscr{H}) \mid\ \dim \operatorname{Ran}(A) < \infty\}}, \tag{2.6}$$

where the closure is taken in the operator norm.

The *Schatten p-classes*,

$$\mathscr{T}_p(\mathscr{H}) = \{A \in \mathscr{C}(\mathscr{H}) \mid \|A\|_p < \infty\}, \tag{2.7}$$

where

$$\|A\|_p = \sup\{(\sum_j |\langle \phi_j, A\psi_j\rangle|^p)^{\frac{1}{p}} \mid \{\phi_j\}, \{\psi_j\} \text{ ONS}\} \tag{2.8}$$

(the supremum over all orthonormal sets) denotes the p-norm of A, are Banach spaces. We have

$$\|A\| \leqslant \|A\|_p. \tag{2.9}$$

The space $\mathscr{T}_1(\mathscr{H})$ is called the space of *trace class* operators. If A is trace class, then the *trace* of A,

$$\operatorname{tr}(A) = \sum_j \langle \varphi_j, A\varphi_j \rangle, \tag{2.10}$$

is finite and independent of the orthonormal basis $\{\phi_j\}$. Moreover, by the Lidskij trace theorem the trace of a trace class operator is the sum over all eigenvalues counted with their multiplicity, see e.g. [43].

Definition 2.6. *We call $P \in \mathscr{L}(\mathscr{H})$ where*

$$P^2 = P \tag{2.11}$$

a projection*. If in addition P is self-adjoint we call P an* orthogonal projection*.*

A projection $P \in \mathscr{L}(\mathscr{H})$ acts like the identity on $\operatorname{Ran}(P)$ which is a closed subspace of $\mathscr{H}$. An orthogonal projection $P \in \mathscr{L}(\mathscr{H})$ moreover acts like the zero operator on $\operatorname{Ran}(P)^\perp$.

Remark 2.7. *For a self-adjoint projection P we have*

$$\dim \operatorname{Ran}(P) = \operatorname{tr}(P) = \|P\|_1. \tag{2.12}$$

If P is not finite-rank, then all three numbers equal ∞.

2.2 Spectra and resolvents

For the herein recalled claims and definitions confer in particular the Sections VI.3, VII.3, and VIII.7 in [33].

Definition 2.8. *Let $A \in \mathscr{L}(\mathscr{H})$. Then, the* resolvent set *$\rho(A)$ of A and the* spectrum *$\sigma(A)$ of A are given by*

$$\rho(A) = \{z \in \mathbb{C} \mid (A - z)^{-1} \in \mathscr{L}(\mathscr{H})\}, \tag{2.13}$$

$$\sigma(H) = \mathbb{C} \setminus \rho(A) \tag{2.14}$$

and the resolvent *of A is the operator-valued function*

$$\begin{aligned} R_A(z) : \rho(A) &\to \mathscr{L}(\mathscr{H}) \\ z &\mapsto (A - z)^{-1}. \end{aligned} \tag{2.15}$$

Now,

$$R_A(z)^* = (A - z)^{*-1} = (A^* - z^*)^{-1} = R_{A^*}(z^*). \tag{2.16}$$

By the inverse mapping theorem (confer e.g. Theorem III.11 in [33]) the inverse of a bounded linear operator from a Banach space *onto* a Banach space is bounded if it exists. Hence, suppose $A \in \mathscr{L}(\mathscr{H})$, then $z \in \rho(A)$ if $A - z$ is bijective. Moreover, $\rho(A)$ is open, $\emptyset \neq \sigma(A) \subseteq \mathcal{B}_{\|A\|}(0)$, and $R_A(z)$ is an analytic $\mathscr{L}(\mathscr{H})$-valued function on each component of $\rho(A)$.

Theorem 2.9. *Confer Theorem 2.23 in [43]. Let A_j be self-adjoint operators on $\mathscr{H}_j$. Then, the countable orthogonal sum $A = \oplus_j A_j$ is self-adjoint,*

$$\sigma(A) = \overline{\cup_j \sigma(A_j)}, \tag{2.17}$$

where the closure can be omitted if there are only finitely many terms, and

$$R_A(z) = \oplus_j R_{A_j}(z) \tag{2.18}$$

holds for all $z \notin \sigma(A)$.

Definition 2.10. *Let $\psi \in \mathscr{H}, \psi \neq 0, z \in \mathbb{C}$, such that*

$$A\psi = z\psi \tag{2.19}$$

holds, then ψ is called an eigenvector *corresponding to the* eigenvalue *z of A. The set of all eigenvalues of A is called the* point spectrum *$\sigma_p(A)$ of A. The* multiplicity *of an eigenvector ψ is the dimension of the corresponding space of eigenvectors. We denote the number of eigenvalues of A in an interval I as $E_I(A)$.*

If z is an eigenvalue of A, then $A - z$ is not injective and hence $\sigma_p(A) \subseteq \sigma(A)$.

Theorem 2.11. *Confer Theorem VI.8 in [33]. If A is self-adjoint, then $\sigma(A) \subseteq \mathbb{R}$ and eigenvectors corresponding to distinct eigenvalues of A are orthogonal.*

Let $P_\Omega(A)$ denote the family of spectral projections associated with a self-adjoint operator A. We have, confer [33, Section VII.3],

$$z \in \sigma(A) \iff P_{(z-\varepsilon, z+\varepsilon)}(A) \neq 0 \quad \text{for all } \varepsilon > 0. \tag{2.20}$$

Definition 2.12. *The* essential spectrum *$\sigma_{ess}(A)$ and the* discrete spectrum *$\sigma_d(A)$ of A are given by*

$$\sigma_{ess}(A) = \{z \in \mathbb{R} \mid \dim \operatorname{Ran} P_{(z-\varepsilon,z+\varepsilon)}(A) = \infty \textit{ for all } \varepsilon > 0\}, \tag{2.21}$$
$$\sigma_d(A) = \{z \in \sigma(A) \mid \dim \operatorname{Ran} P_{(z-\varepsilon,z+\varepsilon)}(A) < \infty \textit{ for some } \varepsilon > 0\}. \tag{2.22}$$

We have

$$\sigma(A) = \sigma_{ess}(A) \cup \sigma_d(A) \quad \text{and} \quad \sigma_{ess}(A) \cap \sigma_d(A) = \emptyset. \tag{2.23}$$

The essential spectrum $\sigma_{ess}(A)$ is closed in $\mathbb{R}$, while $\sigma_d(A)$ is not necessarily closed. We have

$$\sigma_d(A) \subseteq \sigma_p(A) \subseteq \sigma(A). \tag{2.24}$$

Theorem 2.13. *We have*

$$z \in \sigma_d(A) \quad \Longleftrightarrow \quad \begin{cases} z \text{ is a discrete point of } \sigma(A) \text{ and} \\ z \text{ is an eigenvalue of finite multiplicity.} \end{cases} \tag{2.25}$$

Theorem 2.14 (classical Weyl theorem)**.** *Confer [33]. If A is self-adjoint and C is compact, then*

$$\sigma_{ess}(A) = \sigma_{ess}(A + C). \tag{2.26}$$

In the next lemma we apply this theorem to our particular situation. Recall that H and $H_\pm$ are the Jacobi operators introduced in (1.5) and (1.7). Hence, we see that our main assumption $a_0 = a_1$ and $\lim_{|n|\to\infty} b_0(n) = b_1(n)$ ensures that both operators have the same essential spectrum, we even have

Lemma 2.15. *Let $\lim_{|n|\to\infty}(a_0 - a_1)(n) = 0$ and $\lim_{|n|\to\infty}(b_0 - b_1)(n) = 0$, then*

$$\sigma_{ess}(H_0) = \sigma_{ess}(H_1) \quad \text{and} \quad \sigma_{ess}(H_\pm^0) = \sigma_{ess}(H_\pm^1). \tag{2.27}$$

Proof. Consider

$$\begin{aligned} H_1 - H_0 : \ell^2(\mathbb{Z}) &\to \ell^2(\mathbb{Z}) \\ \psi(n) &\mapsto ((\tau_1 - \tau_0)\psi)(n) \end{aligned}$$

and let $(A_k)_{k\in\mathbb{N}}$ be a sequence of finite rank operators such that

$$(A_i\psi)(n) = \begin{cases} ((\tau_1 - \tau_0)\psi)(n) & \text{if } |n| \leqslant k \\ 0 & \text{otherwise.} \end{cases}$$

By

$$\lim_{k\to\infty} \|A_k - (H_1 - H_0)\| = \lim_{k\to\infty} \sup_{\psi | \|\psi\|=1} \|A_k\psi - (H_1 - H_0)\psi\| = 0$$

the operator $H_1 - H_0$ is the norm limit of a sequence of finite rank operators and hence compact. Thus, $\sigma_{ess}(H_0) = \sigma_{ess}(H_0 + H_1 - H_0) = \sigma_{ess}(H_1)$ by the previous theorem. Moreover, $H_\pm^1 - H_\pm^0$ is compact and hence $\sigma_{ess}(H_\pm^0) = \sigma_{ess}(H_\pm^1)$. □

2.3 Operator convergence

The following can be found in Section 9.3 of [49] about operator convergence in norm resolvent and strong resolvent sense.

Definition 2.16. *Let $A_n, A \in \mathscr{L}(\mathscr{H})$. We say A_n converges to A in norm, resp. A_n converges to A strongly,*

$$A_n \to A, \quad \textit{if} \quad \lim_{n\to\infty} \|A_n - A\| = 0, \textit{ resp.} \tag{2.28}$$

$$A_n \xrightarrow{s} A, \quad \textit{if} \quad \lim_{n\to\infty} \|A_n\psi - A\psi\| = 0 \textit{ for all } \psi \in \mathscr{H}. \tag{2.29}$$

Definition 2.17. *Let A_n, A be self-adjoint operators (in Hilbert spaces). We say A_n converges to A in norm resolvent sense, resp. in strong resolvent sense,*

$$A_n \xrightarrow{nr} A, \quad \textit{if} \quad R_z(A_n) \to R_z(A) \textit{ for some } z \in \Gamma, \textit{ resp.} \tag{2.30}$$

$$A_n \xrightarrow{sr} A, \quad \textit{if} \quad R_z(A_n) \xrightarrow{s} R_z(A) \textit{ for some } z \in \Gamma, \tag{2.31}$$

where $\Gamma = \rho(A) \cap (\cap_n \rho(A_n))$.

If A_n converges to A in norm (strong) resolvent sense for some $z \in \Gamma$, then A_n converges to A in norm (strong) resolvent sense for all $z \in \Gamma$.

Theorem 2.18. *Confer Theorem VIII.24 in [33]. Let A_n, A be self-adjoint operators and $A_n \xrightarrow{sr} A$, then*

$$z \in \sigma(A) \quad \Longrightarrow \quad \exists\; z_n \in \sigma(A_n) \textit{ such that } z_n \to z. \tag{2.32}$$

Lemma 2.19. *Confer Lemma 5 in [44]. Let A_n, A, be self-adjoint operators, $z_- < z_+$, and let $A_n \xrightarrow{sr} A$, then*

$$\liminf_{n\to\infty} \operatorname{tr}(P_{(z_-,z_+)}(A_n)) \geqslant \operatorname{tr}(P_{(z_-,z_+)}(A)). \tag{2.33}$$

If moreover

$$\limsup_{n\to\infty} \operatorname{tr}(P_{(z_-,z_+)}(A_n)) \leqslant \operatorname{tr}(P_{(z_-,z_+)}(A)) \tag{2.34}$$

holds, then

$$\lim_{n\to\infty} \operatorname{tr}(P_{(z_-,z_+)}(A_n)) = \operatorname{tr}(P_{(z_-,z_+)}(A)). \tag{2.35}$$

Proof. Equation (2.33) is shown in [19], Lemma 5.2. Clearly, (2.33) and (2.34) imply (2.35). □

Definition 2.20. *Let A be closeable and let $\mathscr{D}_0$ be a linear subspace of $\mathscr{D}(A)$, then we say $\mathscr{D}_0$ is a core of A if $\overline{A|_{\mathscr{D}_0}}$ is an extension of A.*

If A is closed, then $\overline{A|_{\mathscr{D}_0}} = A$. If $A \in \mathscr{L}(\mathscr{H})$, then every dense linear subspace of $\mathscr{H}$ is a core of A. In the case of bounded operators norm (strong) convergence implies norm (strong) resolvent convergence, see [43] and

Theorem 2.21. *See Satz 9.22 in [49]. Let A_n, A be self-adjoint operators in $\mathscr{H}$. Then,*

$$A_n \xrightarrow{sr} A$$

if one of the following conditions holds:

a *there is a core $\mathscr{D}_0$ of A such that to every $\psi \in \mathscr{D}_0$ there exists an $n_0 = n_0(\psi) \in \mathbb{N}$ such that $\psi \in \mathscr{D}(A_n)$ for $n \geqslant n_0$ and $A_n\psi \to A\psi$ as $n \to \infty$.*

b *we have $A_n, A \in \mathscr{L}(\mathscr{H})$ and $A_n \xrightarrow{s} A$.*

2.4 Green function and Weyl solutions

Now we have a closer look at the resolvents of Jacobi matrices, confer [42]. Therefore, let δ_j be the sequence $\delta_j(i) = \delta_{ij}$ where δ_{ij} denotes the Kronecker delta and recall H and $H_\pm$ from (1.5) and (1.7).

Definition 2.22. *For all $z \in \rho(H)$ the resolvent $G(z) = R_H(z) = (H-z)^{-1}$ is given by an infinite matrix*

$$\begin{aligned} G(z) : \ell^2(\mathbb{Z}) &\to \ell^2(\mathbb{Z}) \\ \psi &\mapsto (H-z)^{-1}\psi. \end{aligned} \tag{2.36}$$

The matrix elements of $G(z)$, where the element at the m-th row and the n-th column is denoted by $G(z,m,n) = \langle \delta_m, (H-z)^{-1}\delta_n \rangle$, are called the Green function.

Lemma 2.23. *The Green function fulfills*

$$G(z^*, m, n) = G(z, m, n)^*, \tag{2.37}$$

$$G(z, m, n) = G(z, n, m), \text{ and} \tag{2.38}$$

$$(H-z)G(z, \cdot, n) = \delta_n(\cdot), \tag{2.39}$$

where $G(z,\cdot,n)$ denotes the n-th column of $G(z)$.

Proof. By (2.16) we have

$$G(z)^* = R_H(z)^* = R_{H^*}(z^*) = R_H(z^*) = G(z^*).$$

Let A^T denote the transpose of A. Then, the second claim follows from

$$\mathbb{I} = (H-z)G(z) = ((H-z)G(z))^T = G(z)^T(H-z)^T = G(z)^T(H-z),$$

hence $G(z)^T = G(z)$. For the last claim consider $(H-z)G(z) = \mathbb{I}$. ☐

The next lemma can be found on p. 6 in [42] and moreover follows from (3.5).

Lemma 2.24. *Let $u, \tilde{u}$ be solutions of $\tau u = zu$, then the Wronskian*

$$W_n(u,\tilde{u}) = a(n)(u(n)\tilde{u}(n+1) - u(n+1)\tilde{u}(n))$$

is constant. Moreover, $W(u, \tilde{u})$ vanishes iff u and $\tilde{u}$ are linearly dependent, i.e. there exists an $\alpha \in \mathbb{R}, \alpha \neq 0$, such that $\alpha u = \tilde{u}$.

Lemma 2.25. *If $z \notin \sigma_{ess}(H)$, then there exist solutions*

$$u_{\pm}(z) \in \ell^2(\pm\mathbb{N}) \tag{2.40}$$

of $\tau - z$ which are unique up to a multiple (and square summable near $\pm\infty$). Those solutions are called Weyl solutions*. Moreover, the eigenvalues of H are simple.*

Proof. If $z \notin \sigma_{ess}(H)$, then $z \in \rho(H)$ or $z \in \sigma_d(H)$.
If $z \in \rho(H)$, then the resolvent $G(z)$ exists and all columns (and hence by symmetry all rows) of $G(z)$ are in $\ell^2(\mathbb{Z})$ by Lemma 2.23:

$$G(z, \cdot, n) = R_H(z)\delta_n \in \ell^2(\mathbb{Z}).$$

Those $\phi_n = G(z, \cdot, n)$ are solutions of $\tau - z$ at $j < n$ and $j > n$ by

$$((\tau - z)\phi_n)(j) = 0.$$

Choosing initial values $\phi_n(n+1)$ and $\phi_n(n+2)$ we obtain a solution $u_{+,n}(z) \in \ell(\mathbb{Z})$ of $\tau - z$ which is square summable near ∞. Now, let $u_+(z)$ be another solution of $\tau - z$ in $\ell^2(\mathbb{N})$, then by Lemma 2.24 the Wronskian of $u_{+,n}(z)$ and $u_+(z)$ is constant and hence vanishes by

$$\lim_{n \to \infty} W_n(u_{+,n}(z), u_+(z)) = 0.$$

Thus, by Lemma 2.24 the solution $u_+(z)$ is a constant multiple of $u_{+,n}(z)$. Analogously, we obtain a solution $u_{-,n}(z)$ of $\tau - z$ which is square summable near $-\infty$ by choosing initial conditions $\phi_n(n-1)$ and $\phi_n(n-2)$.
If $z \in \sigma_d(H)$, then z is an eigenvalue of H by (2.24) and hence there exists a solution of $\tau - z$ in $\ell^2(\mathbb{Z})$, namely the corresponding eigensequence $\psi(z)$. Let $u_+(z)$ and $u_-(z)$ be solutions of $\tau - z$ which are square summable near $\pm\infty$ (or an eigensequence of H corresponding to z), then again by Lemma 2.24 and

$$\lim_{n \to \pm\infty} W_n(u_{\pm}(z), \psi(z)) = 0$$

the Weyl solutions are a constant multiple of ψ. Hence, the eigenvalues of H are simple. $\square$

The spectra of H_+ and H_- are also simple, confer therefore Chapter 3 in [42], and we have

$$\sigma_{ess}(H) = \sigma_{ess}(H_+) \cup \sigma_{ess}(H_-). \tag{2.41}$$

Hence, the discrete spectrum of H is the set of all discrete points of $\sigma(H)$ and $\sigma_{ess}(H)$ is the set of all accumulation points of $\sigma(H)$. The same holds for the spectra of H_+ and H_-. And if $[z_-, z_+] \cap \sigma_{ess}(H) = \emptyset$, then $E_{[z_-,z_+]}(H)$ is finite. Now, we state the resolvents explicitly:

Lemma 2.26. *Let $z \in \rho(H)$, then the Green function is given by*

$$G(z, m, n) = W(u_-(z), u_+(z))^{-1} \begin{cases} u_-(z, m)u_+(z, n) & \text{if } m \leqslant n, \\ u_+(z, m)u_-(z, n) & \text{if } m \geqslant n, \end{cases} \tag{2.42}$$

where $u_\pm(z)$ denote the Weyl solutions of $\tau - z$.

Proof. We show that (2.42) fulfills

$$(H - z)G(z) = \mathbb{I}$$

for all entries of $\mathbb{I}$. Therefore, we abbreviate $u_\pm = u_\pm(z)$ and observe that we have

$$\begin{aligned} &a(m-1)G(z, m-1, n) + (b(m) - z)G(z, m, n) + a(m)G(z, m+1, n) \\ &\quad = W(u_-, u_+)^{-1}(-u_+(n)a(n)u_-(n+1) + a(n)u_+(n+1)u_-(n)) = 1. \end{aligned}$$

at the diagonal $(m = n)$. At the upper triangle $(m < n)$ we have

$$\begin{aligned} &a(m-1)G(z, m-1, n) + (b(m) - z)G(z, m, n) + a(m)G(z, m+1, n) \\ &\quad = W(u_-, u_+)^{-1}(u_+(n)(a(m-1)u_-(m-1) \\ &\qquad + (b(m) - z)u_-(m) + a(m)u_-(m+1))) = 0 \end{aligned}$$

and at the lower triangle $(m > n)$ we have

$$\begin{aligned} &a(m-1)G(z, m-1, n) + (b(m) - z)G(z, m, n) + a(m)G(z, m+1, n) \\ &\quad = W(u_-, u_+)^{-1}(u_-(n)(a(m-1)u_+(m-1) \\ &\qquad + (b(m) - z)u_+(m) + a(m)u_+(m+1))) = 0. \end{aligned}$$

□

Now, look at the following Jacobi matrices with variable base points: let

$$H_{\mathsf{m},\mathsf{n}} = \begin{pmatrix} b(\mathsf{m}+1) & a(\mathsf{m}+1) & & \\ a(\mathsf{m}+1) & b(\mathsf{m}+2) & \ddots & \\ & \ddots & \ddots & a(\mathsf{n}-2) \\ & & a(\mathsf{n}-2) & b(\mathsf{n}-1) \end{pmatrix} \tag{2.43}$$

be the finite Jacobi matrix with base points m, n (which we'll omit whenever a base point equals 0) in $\ell(\mathsf{m},\mathsf{n}) = \ell(\{n \in \mathbb{Z} \mid \mathsf{m} < n < \mathsf{n}\})$, where $\mathsf{n} - \mathsf{m} > 2$. And analogously let

$$\begin{aligned} &H_{\mathsf{m},+} : \ell^2(\mathsf{m},\infty) \to \ell^2(\mathsf{m},\infty) \\ &\qquad (H_{\mathsf{m},+}\psi)(n) = \begin{cases} b(n)\psi(n) + a(n)\psi(n+1) & \text{if } n = \mathsf{m}+1 \\ (\tau\psi)(n) & \text{if } n > \mathsf{m}+1 \end{cases} \end{aligned} \tag{2.44}$$

be a Jacobi matrix in the upper half-line and

$$\begin{aligned} &H_{-,\mathsf{n}} : \ell^2(-\infty,\mathsf{n}) \to \ell^2(-\infty,\mathsf{n}) \\ &\qquad (H_{-,\mathsf{n}}\psi)(n) = \begin{cases} b(n)\psi(n) + a(n-1)\psi(n-1) & \text{if } n = \mathsf{n}-1 \\ (\tau\psi)(n) & \text{if } n < \mathsf{n}-1 \end{cases} \end{aligned} \tag{2.45}$$

a Jacobi matrix in the lower half-line.

Lemma 2.27. *Fix $\mathsf{m} \in \mathbb{Z}$ and let $z \in \rho(H_{\mathsf{m},+})$, then the Green function is given by*

$$G_{\mathsf{m},+}(z,m,n) = W(\psi_{\mathsf{m}}(z), u_+(z))^{-1} \begin{cases} \psi_{\mathsf{m}}(z,m)u_+(z,n) & \text{if } m \leqslant n, \\ u_+(z,m)\psi_{\mathsf{m}}(z,n) & \text{if } n \leqslant m, \end{cases} \tag{2.46}$$

where $u_+(z)$ is a Weyl solution of $\tau - z$ and $\psi_{\mathsf{m}}(z)$ denotes a solution fulfilling $\psi_{\mathsf{m}}(z,\mathsf{m}) = 0$.

Proof. We show that (2.46) fulfills $(H_{\mathsf{m},+} - z)G_{\mathsf{m},+}(z) = \mathbb{I}$ for all entries of $\mathbb{I}$. Abbreviate $u_+ = u_+(z)$, then, at the first entry ($m = \mathsf{m}+1 = n$) we have

$$\begin{aligned} &(b(m) - z)G_{\mathsf{m},+}(m,n) + a(m)G_{\mathsf{m},+}(m+1,n) \\ &\quad = W(\psi_{\mathsf{m}}, u_+)^{-1}(\psi_{\mathsf{m}}(n)((b(m)-z)u_+(m) + a(m)u_+(m+1))) \\ &\quad = -W(s,u_+)^{-1}a(\mathsf{m})u_+(\mathsf{m})\psi_{\mathsf{m}}(\mathsf{m}+1) = 1 \end{aligned}$$

and at the rest of the first row ($m = \mathsf{m}+1 < n$) we have

$$\begin{aligned} &(b(m) - z)G_{\mathsf{m},+}(m,n) + a(m)G_{\mathsf{m},+}(m+1,n) \\ &\quad = \frac{u_+(n)}{W(\psi_{\mathsf{m}}, u_+)}((b(\mathsf{m}+1) - z)\psi_{\mathsf{m}}(\mathsf{m}+1) + a(\mathsf{m}+1)\psi_{\mathsf{m}}(\mathsf{m}+2)) = 0. \end{aligned}$$

Now, consider all other rows of $\mathbb{I}$, that is $m > \mathsf{m}+1$. Then,

$$\begin{aligned} &a(m-1)G_{\mathsf{m},+}(m-1,n) + (b(m)-z)G_{\mathsf{m},+}(m,n) + a(m)G_{\mathsf{m},+}(m+1,n) \\ &\quad = W(\psi_{\mathsf{m}}, u_+)^{-1}(a(n)u_+(n+1)\psi_{\mathsf{m}}(n) - u_+(n)a(n)\psi_{\mathsf{m}}(n+1)) = 1 \end{aligned}$$

at the diagonal ($\mathsf{m}+1 < m = n$) and

$$\begin{aligned} &a(m-1)G_{\mathsf{m},+}(m-1,n) + (b(m)-z)G_{\mathsf{m},+}(m,n) + a(m)G_{\mathsf{m},+}(m+1,n) \\ &\quad = W(\psi_{\mathsf{m}}, u_+)^{-1}(u_+(n)(a(m-1)\psi_{\mathsf{m}}(m-1) \\ &\qquad + (b(m)-z)\psi_{\mathsf{m}}(m) + a(m)\psi_{\mathsf{m}}(m+1))) = 0 \end{aligned}$$

at the upper triangle ($\mathsf{m}+1 < m < n$). At the lower triangle ($\mathsf{m}+1 < m > n$)

$$\begin{aligned} &a(m-1)G_{\mathsf{m},+}(m-1,n) + (b(m)-z)G_{\mathsf{m},+}(m,n) + a(m)G_{\mathsf{m},+}(m+1,n) \\ &\quad = W(\psi_{\mathsf{m}}, u_+)^{-1}(\psi_{\mathsf{m}}(n)(a(m-1)u_+(m-1) \\ &\qquad + (b(m)-z)u_+(m) + a(m)u_+(m+1))) = 0 \end{aligned}$$

holds. □

Analogously we find

Lemma 2.28. *Fix* $\mathsf{n} \in \mathbb{Z}$ *and let* $z \in \rho(H_{-,\mathsf{n}})$*, then the Green function is given by*

$$G_{-,\mathsf{n}}(z,m,n) = -W(\psi_{\mathsf{n}}(z), u_-(z))^{-1} \begin{cases} \psi_{\mathsf{n}}(z,n)u_-(z,m) & \text{if } m \leqslant n, \\ u_-(z,n)\psi_{\mathsf{n}}(z,m) & \text{if } n \leqslant m, \end{cases}$$

where $u_-(z)$ *is a Weyl solution of* $\tau - z$ *and* $\psi_{\mathsf{n}}(z)$ *denotes a solution fulfilling* $\psi_{\mathsf{n}}(z,\mathsf{n}) = 0$.

Lemma 2.29. *Fix* m, n *and let* $z \in \rho(H_{\mathsf{m},\mathsf{n}})$*, then the Green function is given by*

$$G_{\mathsf{m},\mathsf{n}}(z,m,n) = W(\psi_{\mathsf{m}}(z), \psi_{\mathsf{n}}(z))^{-1} \begin{cases} \psi_{\mathsf{m}}(z,m)\psi_{\mathsf{n}}(z,n) & \text{if } m \leqslant n, \\ \psi_{\mathsf{n}}(z,m)\psi_{\mathsf{m}}(z,n) & \text{if } m \geqslant n, \end{cases}$$

where $\psi_{\mathsf{m}}(z)$ *is a solution fulfilling* $\psi_{\mathsf{m}}(z,\mathsf{m}) = 0$ *and* $\psi_{\mathsf{n}}(z)$ *is a solution fulfilling* $\psi_{\mathsf{n}}(z,\mathsf{n}) = 0$.

Proof. At the first row ($m = \mathsf{m}+1$) we have

$$(b(\mathsf{m}+1)-z)G_{\mathsf{m},\mathsf{n}}(\mathsf{m}+1,n) + a(\mathsf{m}+1)G_{\mathsf{m},\mathsf{n}}(\mathsf{m}+2,n) = \delta_{n,\mathsf{m}+1}$$

by

$$\begin{aligned} &(b(\mathsf{m}+1)-z)G_{\mathsf{m},\mathsf{n}}(\mathsf{m}+1,\mathsf{m}+1) + a(\mathsf{m}+1)G_{\mathsf{m},\mathsf{n}}(\mathsf{m}+2,\mathsf{m}+1) \\ &\quad = (a(\mathsf{m})\psi_{\mathsf{m}}(\mathsf{m}+1)\psi_{\mathsf{n}}(\mathsf{m}))^{-1}\psi_{\mathsf{m}}(\mathsf{m}+1)a(\mathsf{m})\psi_{\mathsf{n}}(\mathsf{m}) = 1 \end{aligned}$$

and

$$\begin{aligned}
&(b(\mathsf{m}+1)-z)G_{\mathsf{m},\mathsf{n}}(\mathsf{m}+1,n)+a(\mathsf{m}+1)G_{\mathsf{m},\mathsf{n}}(\mathsf{m}+2,n)\\
&\quad= W(\psi_{\mathsf{m}},\psi_{\mathsf{n}})^{-1}((b(\mathsf{m}+1)-z)\psi_{\mathsf{m}}(\mathsf{m}+1)\psi_{\mathsf{n}}(n)\\
&\qquad+a(\mathsf{m}+1)\psi_{\mathsf{m}}(\mathsf{m}+2)\psi_{\mathsf{n}}(n))\\
&\quad= -W(\psi_{\mathsf{m}},\psi_{\mathsf{n}})^{-1}\psi_{\mathsf{n}}(n)a(\mathsf{m})\psi_{\mathsf{m}}(\mathsf{m})=0
\end{aligned}$$

if $n > \mathsf{m}+1$. At the last row ($m = \mathsf{n}-1$) we have

$$a(\mathsf{n}-2)G_{\mathsf{m},\mathsf{n}}(\mathsf{n}-2,n)+(b(\mathsf{n}-1)-z)G_{\mathsf{m},\mathsf{n}}(\mathsf{n}-1,n)=\delta_{n,\mathsf{n}-1}$$

and in between ($\mathsf{m}+2 \leqslant m \leqslant \mathsf{n}-2$) we have

$$a(m-1)G_{\mathsf{m},\mathsf{n}}(m-1,n)+(b(m)-z)G_{\mathsf{m},\mathsf{n}}(m,n)+a(m)G_{\mathsf{m},\mathsf{n}}(m+1,n)=\delta_{m,n}.$$

Thus, $(H_{\mathsf{m},\mathsf{n}}-z)G_{\mathsf{m},\mathsf{n}}(z)=\mathbb{I}$. □

2.5 Weyl m-functions

Finally, the concept of Weyl m-functions for Jacobi operators is briefly recalled. We use this concept in Section 8.3 which in turn is necessary for the proof of our main theorem *above* the infimum of the essential spectrum.

Definition 2.30. *Let z be in the respective resolvent set, that is, $z \in \rho(H_{\mathsf{m},+})$, $z \in \rho(H_{\mathsf{m},\mathsf{n}})$, or $z \in \rho(H_{-,\mathsf{n}})$. Then,*

$$m_+(z,\mathsf{m}) = \langle\delta_{\mathsf{m}+1},(H_{\mathsf{m},+}-z)^{-1}\delta_{\mathsf{m}+1}\rangle = G_{\mathsf{m},+}(z,\mathsf{m}+1,\mathsf{m}+1), \tag{2.47}$$
$$m_-(z,\mathsf{n}) = \langle\delta_{\mathsf{n}-1},(H_{-,\mathsf{n}}-z)^{-1}\delta_{\mathsf{n}-1}\rangle = G_{-,\mathsf{n}}(z,\mathsf{n}-1,\mathsf{n}-1), \tag{2.48}$$
$$m_+^{\mathsf{n}}(z,\mathsf{m}) = \langle\delta_{\mathsf{m}+1},(H_{\mathsf{m},\mathsf{n}}-z)^{-1}\delta_{\mathsf{m}+1}\rangle = G_{\mathsf{m},\mathsf{n}}(z,\mathsf{m}+1,\mathsf{m}+1), \tag{2.49}$$
$$m_-^{\mathsf{m}}(z,\mathsf{n}) = \langle\delta_{\mathsf{n}-1},(H_{\mathsf{m},\mathsf{n}}-z)^{-1}\delta_{\mathsf{n}-1}\rangle = G_{\mathsf{m},\mathsf{n}}(z,\mathsf{n}-1,\mathsf{n}-1) \tag{2.50}$$

are the Weyl m-functions*.*

We already know from our previous considerations that the Weyl m-function can be expressed in terms of solutions fullfilling the right/left boundary condition of the corresponding operator:

Lemma 2.31. *If z is in the respective resolvent set, $\rho(H_{\mathsf{m},+}),\rho(H_{\mathsf{m},\mathsf{n}})$, or $\rho(H_{-,\mathsf{n}})$, then*

$$m_+(z,\mathsf{m}) = -\frac{u_+(z,\mathsf{m}+1)}{a(\mathsf{m})u_+(z,\mathsf{m})}, \qquad |m_+(z,\mathsf{m})| \leqslant \|(H_{\mathsf{m},+}-z)^{-1}\|,$$
$$m_-(z,\mathsf{n}) = -\frac{u_-(z,\mathsf{n}-1)}{a(\mathsf{n}-1)u_-(z,\mathsf{n})}, \qquad |m_-(z,\mathsf{n})| \leqslant \|(H_{-,\mathsf{n}}-z)^{-1}\|,$$

$$m_+^{\mathsf{n}}(z,\mathsf{m}) = -\frac{\psi_{\mathsf{n}}(z,\mathsf{m}+1)}{a(\mathsf{m})\psi_{\mathsf{n}}(z,\mathsf{m})}, \qquad |m_+^{\mathsf{n}}(z,\mathsf{m})| \leqslant \|(H_{\mathsf{m},\mathsf{n}}-z)^{-1}\|,$$
$$m_-^{\mathsf{m}}(z,\mathsf{n}) = -\frac{\psi_{\mathsf{m}}(z,\mathsf{n}-1)}{a(\mathsf{n}-1)\psi_{\mathsf{m}}(z,\mathsf{n})}, \qquad |m_-^{\mathsf{m}}(z,\mathsf{n})| \leqslant \|(H_{\mathsf{m},\mathsf{n}}-z)^{-1}\|.$$

Proof. By Lemma 2.27 and $\psi_{\mathsf{m}}(z,\mathsf{m}) = 0$

$$\begin{aligned} m_+(z,\mathsf{m}) &= G_{\mathsf{m},+}(z,\mathsf{m}+1,\mathsf{m}+1) \\ &= W(\psi_{\mathsf{n}}(z),u_+(z))^{-1}\psi_{\mathsf{m}}(z,\mathsf{m}+1)u_+(z,\mathsf{m}+1) \\ &= -\frac{u_+(z,\mathsf{m}+1)}{a(\mathsf{m})u_+(z,\mathsf{m})} \end{aligned}$$

holds and by Lemma 2.28 and $\psi_{\mathsf{n}}(z,\mathsf{n}) = 0$ we have

$$\begin{aligned} m_-(z,\mathsf{n}) &= G_{-,\mathsf{n}}(z,\mathsf{n}-1,\mathsf{n}-1) \\ &= -W(\psi_{\mathsf{n}}(z),u_-(z))^{-1}\psi_{\mathsf{n}}(z,\mathsf{n}-1)u_-(z,\mathsf{n}-1) \\ &= -\frac{u_-(z,\mathsf{n}-1)}{a(\mathsf{n}-1)u_-(z,\mathsf{n})}. \end{aligned}$$

By Lemma 2.29 and $s_-^{\mathsf{m}}(z,\mathsf{m}) = 0$

$$\begin{aligned} m_+^{\mathsf{n}}(z,\mathsf{m}) &= G_{\mathsf{m},\mathsf{n}}(z,\mathsf{m}+1,\mathsf{m}+1) \\ &= W(\psi_{\mathsf{m}}(z),\psi_{\mathsf{n}}(z))^{-1}\psi_{\mathsf{m}}(z,\mathsf{m}+1)\psi_{\mathsf{n}}(z,\mathsf{m}+1) \\ &= -\frac{\psi_{\mathsf{n}}(z,\mathsf{m}+1)}{a(\mathsf{m})\psi_{\mathsf{n}}(z,\mathsf{m})} \end{aligned}$$

holds and by $s_-^{\mathsf{n}}(z,\mathsf{n}) = 0$ we have

$$\begin{aligned} m_-^{\mathsf{m}}(z,\mathsf{n}) &= G_{\mathsf{m},\mathsf{n}}(z,\mathsf{n}-1,\mathsf{n}-1) \\ &= W(\psi_{\mathsf{m}}(z),\psi_{\mathsf{n}}(z))^{-1}\psi_{\mathsf{m}}(z,\mathsf{n}-1)\psi_{\mathsf{n}}(z,\mathsf{n}-1) \\ &= -\frac{\psi_{\mathsf{m}}(z,\mathsf{n}-1)}{a(\mathsf{n}-1)\psi_{\mathsf{m}}(z,\mathsf{n})}. \end{aligned}$$

□

If we have strong resolvent convergence, then of course also the corresponding Weyl m-functions converge (provided the resolvents exist):

Lemma 2.32. *Fix some* $\mathsf{m} \in \mathbb{Z}$. *If, as* $n \to \infty$, $m_+^n(z,\mathsf{m})$ *correspond to a sequence of Jacobi matrices* J_n *in* $\ell(\mathsf{m},n)$ *such that* $J_n \oplus \lambda\mathbb{I} \overset{sr}{\to} H_{\mathsf{m},+}$, *then*

$$\lim_{n\to\infty} m_+^n(z,\mathsf{m}) = m_+(z,\mathsf{m}) \tag{2.51}$$

for all $z \in \rho(H_{\mathsf{m},+}) \cap_n \rho(J_n)$ *where* $z \neq \lambda$.

Now, fix some $\mathsf{n} \in \mathbb{Z}$*. If, as* $m \to -\infty$*,* $m_-^m(z, \mathsf{n})$ *correspond to a sequence of Jacobi matrices* J_m *in* $\ell(m, \mathsf{n})$ *such that* $\lambda\mathbb{I} \oplus J_m \stackrel{sr}{\to} H_{-,\mathsf{n}}$*, then*

$$\lim_{m \to -\infty} m_-^m(z, \mathsf{n}) = m_-(z, \mathsf{n}) \tag{2.52}$$

for all $z \in \rho(H_{-,\mathsf{n}}) \cap_m \rho(J_m)$ *where* $z \neq \lambda$*.*

Proof. By $R_{J_n \oplus \lambda\mathbb{I}}(z) = R_{J_n}(z) \oplus R_{\lambda\mathbb{I}}(z)$

$$\begin{aligned}
\lim_{n \to \infty} m_+^n(z, \mathsf{m}) &= \lim_{n \to \infty} \langle \delta_{\mathsf{m}+1}, (J_n - z)^{-1} \delta_{\mathsf{m}+1} \rangle \\
&= \lim_{n \to \infty} \langle \delta_{\mathsf{m}+1}, (J_n \oplus \lambda\mathbb{I} - z)^{-1} \delta_{\mathsf{m}+1} \rangle \\
&= \langle \delta_{\mathsf{m}+1}, (H_{\mathsf{m},+} - z)^{-1} \delta_{\mathsf{m}+1} \rangle = m_+(z, \mathsf{m}).
\end{aligned}$$

holds and by $R_{\lambda\mathbb{I} \oplus J_m}(z) = R_{\lambda\mathbb{I}}(z) \oplus R_{J_m}(z)$ we have

$$\begin{aligned}
\lim_{m \to -\infty} m_-^m(z, \mathsf{n}) &= \lim_{m \to -\infty} \langle \delta_{\mathsf{n}-1}, (J_m - z)^{-1} \delta_{\mathsf{n}-1} \rangle \\
&= \lim_{m \to -\infty} \langle \delta_{\mathsf{n}-1}, (\lambda\mathbb{I} \oplus J_m - z)^{-1} \delta_{\mathsf{n}-1} \rangle \\
&= \langle \delta_{\mathsf{n}-1}, (H_{-,n} - z)^{-1} \delta_{\mathsf{n}-1} \rangle = m_-(z, \mathsf{n}).
\end{aligned}$$

□

Chapter 3

Weighted nodes

In this chapter we introduce the Wronski determinant and its basic properties, in particular the 'derivative' along the $\mathbb{Z}$-axis, see (3.5). We then recall some facts about the Prüfer transformation of solutions of Jacobi difference equations, confer e.g. [42], which we extend (in the last section) to a detailed investigation of the difference Δ of two Prüfer angles. We show that Δ counts the number of nodes of the introduced Wronskian which extends the considerations from [4] to the present more general case.

3.1 Wronskian

At first we look at the Wronskian and establish a few formulas which will be very helpful in the sequel.

Definition 3.1. *Let $\mathbb{D}$ denote the space of difference equations. We define the (modified)* Wronskian *or* Casorati determinant *as*

$$\begin{aligned} W &: \mathbb{D}^2 \times \ell(\mathbb{Z})^2 \to \ell(\mathbb{Z}) \qquad (3.1)\\ (\tau_0, \tau_1, \varphi, \psi) &\mapsto W^{\tau_0,\tau_1}(\varphi, \psi) = (W_n^{\tau_0,\tau_1}(\varphi, \psi))_{n\in\mathbb{Z}} \\ &\qquad = (\varphi(n)a_1(n)\psi(n+1) - \psi(n)a_0(n)\varphi(n+1))_{n\in\mathbb{Z}} \\ &\qquad = \left(\begin{vmatrix} \varphi(n) & \psi(n) \\ a_0(n)\varphi(n+1) & a_1(n)\psi(n+1) \end{vmatrix} \right)_{n\in\mathbb{Z}}. \end{aligned}$$

This definition generalizes the one from [4] to different a's. The corresponding difference equations will be evident from the context and thus we'll abbreviate $W(\varphi, \psi) = W^{\tau_0,\tau_1}(\varphi, \psi)$. The Wronskian has the following properties:

- $W^{\tau_0,\tau_0}(\varphi, \varphi)$ vanishes
- $W^{\tau_0,\tau_1}(\varphi, \psi) = -W^{\tau_1,\tau_0}(\psi, \varphi)$

- $W^{\tau_0,\tau_1}(c\,\varphi,\psi) = W^{\tau_0,\tau_1}(\varphi, c\,\psi) = c\,W^{\tau_0,\tau_1}(\varphi,\psi)$
- $W^{\tau_0,\tau_1}(\varphi+\tilde{\varphi},\psi) = W^{\tau_0,\tau_1}(\varphi,\psi) + W^{\tau_0,\tau_1}(\tilde{\varphi},\psi)$
- $W^{\tau_0,\tau_1}(\varphi,\psi+\tilde{\psi}) = W^{\tau_0,\tau_1}(\varphi,\psi) + W^{\tau_0,\tau_1}(\varphi,\tilde{\psi})$

for all $c \in \mathbb{R}$ and $\varphi, \tilde{\varphi}, \psi, \tilde{\psi} \in \ell(\mathbb{Z})$. From now on we abbreviate

$$\Delta a = a_0 - a_1 \quad \text{and} \quad \Delta b = b_0 - b_1. \tag{3.2}$$

Lemma 3.2 (Green's formula)**.** *We find*

$$\sum_{j=n}^{m}(\varphi(\tau_1\psi) - \psi(\tau_0\varphi))(j) = W_m(\varphi,\psi) - W_{n-1}(\varphi,\psi) \tag{3.3}$$
$$- \sum_{j=n-1}^{m-1} \Delta a(j)(\varphi(j+1)\psi(j) + \varphi(j)\psi(j+1)) - \sum_{j=n}^{m} \Delta b(j)\varphi(j)\psi(j).$$

Proof. We have

$$\begin{aligned}
&\sum_{j=n}^{m}(\varphi(\tau_1\psi) - \psi(\tau_0\varphi))(j) \\
&\quad = \sum_{j=n}^{m}[a_1(j-1)\psi(j-1)\varphi(j) + b_1(j)\psi(j)\varphi(j) + a_1(j)\psi(j+1)\varphi(j)) \\
&\qquad - a_0(j-1)\varphi(j-1)\psi(j) - b_0(j)\varphi(j)\psi(j) - a_0(j)\varphi(j+1)\psi(j))] \\
&\quad = \sum_{j=n}^{m}[a_1(j-1)\psi(j-1)\varphi(j) - a_0(j-1)\varphi(j-1)\psi(j)] \\
&\qquad + \sum_{j=n}^{m}[a_1(j)\psi(j+1)\varphi(j)) - a_0(j)\varphi(j+1)\psi(j) - \Delta b(j)\psi(j)\varphi(j))] \\
&\quad = - \sum_{j=n-1}^{m-1} \Delta a(j)(\varphi(j)\psi(j+1) + \varphi(j+1)\psi(j)) - \sum_{j=n}^{m} \Delta b(j)\psi(j)\varphi(j)) \\
&\qquad - a_1(n-1)\psi(n)\varphi(n-1) + a_0(n-1)\varphi(n)\psi(n-1) \\
&\qquad + a_1(m)\psi(m+1)\varphi(m) - a_0(m)\varphi(m+1)\psi(m).
\end{aligned}$$

□

In particular this 'derivative' is the key ingrediant of many of our forthcoming observations.

Corollary 3.3. *Let* $(\tau_j - z)u_j = 0$*,* $j = 0, 1$*, then*

$$W_m(u_0, u_1) - W_{n-1}(u_0, u_1) \tag{3.4}$$

$$= \sum_{j=n-1}^{m-1} \Delta a(j)(u_0(j+1)u_1(j) + u_0(j)u_1(j+1)) + \sum_{j=n}^{m} \Delta b(j)u_0(j)u_1(j),$$

for all $m \geqslant n$ *and*

$$W_n(u_0, u_1) - W_{n-1}(u_0, u_1) \tag{3.5}$$
$$= \Delta a(n-1)(u_0(n)u_1(n-1) + u_0(n-1)u_1(n)) + \Delta b(n)u_0(n)u_1(n).$$

Hence, if u and $\tilde{u}$ solve $\tau u = zu$, then $W(u, \tilde{u})$ is constant (and vanishes iff u and $\tilde{u}$ are linearly dependent), confer Lemma 2.24.

Lemma 3.4. *Let* $(\tau_j - z)u_j = 0$, $j = 0, 1$, *and* $\underline{u}_j = (u_j, u_j^+) \in \ell(\mathbb{Z}, \mathbb{R}^2)$, *then*

$$W_{n+1}(u_0, u_1) - W_n(u_0, u_1) = \langle \underline{u}_0(n), \begin{pmatrix} 0 & \Delta a(n) \\ \Delta a(n) & \Delta b(n+1) \end{pmatrix} \underline{u}_1(n) \rangle. \tag{3.6}$$

Proof. By (3.5) we have

$$\begin{aligned}
&\langle \begin{pmatrix} u_0(n) \\ u_0(n+1) \end{pmatrix}, \begin{pmatrix} 0 & \Delta a(n) \\ \Delta a(n) & \Delta b(n+1) \end{pmatrix} \begin{pmatrix} u_1(n) \\ u_1(n+1) \end{pmatrix} \rangle \\
&= \langle \begin{pmatrix} u_0(n) \\ u_0(n+1) \end{pmatrix}, \begin{pmatrix} \Delta a(n)u_1(n+1) \\ \Delta a(n)u_1(n) + \Delta b(n+1)u_1(n+1) \end{pmatrix} \rangle \\
&= W_{n+1}(u_0, u_1) - W_n(u_0, u_1).
\end{aligned}$$

□

Note that alternatively another definition for the Wronskian could be used which we now mention briefly. And further, in the appendix we will use it to simplify a few of the computations.

Remark 3.5. *Consider*

$$\begin{aligned}
M : \mathbb{D}^2 \times \ell(\mathbb{Z})^2 &\to \ell(\mathbb{Z}) \\
(\tau_0, \tau_1, \phi, \psi) &\mapsto M^{\tau_0, \tau_1}(\phi, \psi),
\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}
M_n^{\tau_0, \tau_1}(\phi, \psi) &= \phi(n)a_0(n)\psi(n+1) - \psi(n)a_1(n)\phi(n+1) \\
&= \begin{vmatrix} a_0(n)\phi(n) & a_1(n)\psi(n) \\ \phi(n+1) & \psi(n+1) \end{vmatrix}.
\end{aligned} \tag{3.8}$$

Then, for all $n \leqslant m$, *we have*

$$W^{\tau_0, \tau_1}(\phi, \psi) = M^{\tau_1, \tau_0}(\phi, \psi), \tag{3.9}$$

$$\sum_{j=n}^{m}(\phi(\tau_1\psi) - \psi(\tau_0\phi))(j) = \sum_{j=n}^{m}(W_j^{0,1}(\phi,\psi) - M_{j-1}^{0,1}(\phi,\psi) - \Delta b(j)\phi(j)\psi(j)),$$

and, if $\tau_j u_j = 0$, $j = 0, 1$, *then*

$$\sum_{j=n}^{m} W_j(u_0, u_1) = \sum_{j=n}^{m}(M_{j-1}(u_0, u_1) + \Delta b(j)u_0(j)u_1(j)) \tag{3.10}$$

and

$$W_n(u_0, u_1) = M_{n-1}(u_0, u_1) + \Delta b(n)u_0(n)u_1(n). \tag{3.11}$$

The following two lemmas will be very helpful in the sequel, in particular in the approximation as well as for our considerations on finite-rank perturbations.

Lemma 3.6. *Let* $a = a_0 = a_1$ *and* $\phi, \psi \in \ell^2(\pm\mathbb{N})$, *then*

$$W(\phi, \psi) \in \ell^1(\pm\mathbb{N}) \subseteq \ell^2(\pm\mathbb{N}). \tag{3.12}$$

Proof. Let $\phi^+(n) = \phi(n+1)$ and $\psi^+(n) = \psi(n+1)$, then the component-wise products $\phi\psi^+, \psi\phi^+ \in \ell^1(\pm\mathbb{N})$ are summable by Hölder's inequality and we further have $W(\phi, \psi) = a(\phi\psi^+ - \psi\phi^+) \in \ell^1(\pm\mathbb{N})$ by $a \in \ell^\infty(\mathbb{N})$. □

Lemma 3.7. *Let* $u_j(\lambda_j), j = 0, 1$, *be solutions of* $(\tau_j - \lambda_j)u_j(\lambda_j) = 0$ *where* $a_0 = a_1$. *Then,*

$$\begin{aligned} &W_j(u_0(\lambda_0), u_1(\lambda_1)) = 0 \text{ for all } j = m, \dots, n \\ &\quad\iff \exists\, \alpha \neq 0 : u_0(\lambda_0, j) = \alpha u_1(\lambda_1, j) \text{ for all } j = m, \dots, n+1. \end{aligned} \tag{3.13}$$

If so, then either

$$b_0(j) - \lambda_0 = b_1(j) - \lambda_1 \quad \text{or} \quad u_0(j) = u_1(j) = 0 \tag{3.14}$$

holds for all $j = m + 1, \dots, n$.

Proof. By $W_j(u_0, u_1) = a(j)(u_0(j)u_1(j+1) - u_1(j)u_0(j+1)) = 0$ for all $j = m, \dots, n$, we have $u_0(j) = 0 \iff u_1(j) = 0$ for all $j = m, \dots, n+1$. Moreover, by

$$W_j(u_0, u_1) - W_{j-1}(u_0, u_1) = (b_0(j) - \lambda_0 - (b_1(j) - \lambda_1))u_0(j)u_1(j) = 0 \tag{3.15}$$

we have either

$$b_0(j) - \lambda_0 = b_1(j) - \lambda_1 \quad \text{or} \quad u_0(j) = u_1(j) = 0 \tag{3.16}$$

for all $j = m+1, \dots, n$. Without loss, let $u_0(m) \neq 0$, then $u_0(m) = \alpha u_1(m)$ and $u_0(m+1) = \alpha u_1(m+1)$ where $\alpha = \frac{u_0(m)}{u_1(m)}$ by $W_m(u_0, u_1) = 0$. The inductive step: by (3.14) we have

$$\begin{aligned} u_0(j+1) &= -a(j)^{-1}(a(j-1)u_0(j-1) + (b_0(j) - \lambda_0)u_0(j)) \\ &= -a(j)^{-1}(a(j-1)\alpha u_1(j-1) + (b_1(j) - \lambda_1)\alpha u_1(j)) = \alpha u_1(j+1) \end{aligned}$$

for all $j = m+1, \dots, n$, hence the solutions are linearly dependent. □

3.2 Discrete Prüfer transformation

Now the discrete Prüfer transformation will be introduced. Therefore at first recall that n is a *node* (sign-change) of u if

$$u(n) = 0 \quad \text{or} \quad u(n)u(n+1) < 0 \tag{3.17}$$

and as usual we call τ (and also u) oscillatory if one (and hence all) solutions of $\tau u = 0$ have infinitely many nodes. The *number of nodes of u between m and l*, $\#_{(m,l)}(u)$, is the number of nodes n of u where either $m < n < l$ or $n = m$ and $u(m) \neq 0$ holds.

Remark 3.8. *The number of nodes of u doesn't change if we drop the zeros in the sequence u (which is sometimes done in the literature) or replace them by any other value, since, as we will see, any solution u of $\tau u = zu$ changes its sign around zeros. Of course the nodes then appear at other positions.*

Lemma 3.9. *Let u be a solution of (1.1) and $u(n) = 0$, then*

$$u(n-1)u(n+1) < 0. \tag{3.18}$$

Proof. Since all zeros of u are simple

$$u(n+1) = \underbrace{-a(n)^{-1}}_{>0}(\underbrace{a(n-1)}_{<0}u(n-1) + \underbrace{(b(n) - z)u(n)}_{=0}) \neq 0$$

holds. □

Thus, by $(u(n), u(n+1)) \neq (0,0)$ for all $n \in \mathbb{Z}$, the *Prüfer variables* $\rho_u, \theta_u \in \ell(\mathbb{Z})$ are well-defined: let

$$\begin{aligned} u(n) &= \rho_u(n) \sin \theta_u(n), \\ -a(n)u(n+1) &= \rho_u(n) \cos \theta_u(n), \end{aligned} \tag{3.19}$$

so that $\rho_u > 0$, fix $\theta_u(n_0) \in (-\pi, \pi]$ at the initial position n_0, and assume

$$\lceil \theta_u(n)/\pi \rceil \leqslant \lceil \theta_u(n+1)/\pi \rceil \leqslant \lceil \theta_u(n)/\pi \rceil + 1 \tag{3.20}$$

for all $n \in \mathbb{Z}$, then both sequences are well-defined and unique.
As in [29] we also use the slightly refined (compared to [4, 42, 46]) definition of Prüfer variables by taking the secondary diagonals a into account. By $-a > 0$ this will not influence the herein recalled well-known claims on the nodes of solutions, but it simplifies our calculations as soon as we look at the nodes of the Wronskian.
From now on let u be a solution of τ and $\rho, \theta \in \ell(\mathbb{Z})$ be the corresponding Prüfer variables.

Lemma 3.10. *Fix some $n \in \mathbb{Z}$, then there exists some $k \in \mathbb{Z}$ such that*

$$\theta(n) = k\pi + \gamma, \quad \theta(n+1) = k\pi + \Gamma, \tag{3.21}$$

where

$$\gamma \in (0, \frac{\pi}{2}], \quad \Gamma \in (0, \pi] \quad \Longleftrightarrow \quad n \text{ is not a node of } u, \tag{3.22}$$

$$\gamma \in (\frac{\pi}{2}, \pi], \quad \Gamma \in (\pi, 2\pi) \quad \Longleftrightarrow \quad n \text{ is a node of } u \tag{3.23}$$

holds. Moreover,

$$\theta(n) = k\pi + \frac{\pi}{2} \quad \Longleftrightarrow \quad \theta(n+1) = (k+1)\pi. \tag{3.24}$$

Proof. Choose $k \in \mathbb{Z}$ such that $\theta(n) = k\pi + \gamma$, $\gamma \in (0, \pi]$ holds. By (3.20) we have $\Gamma \in (0, 2\pi]$. If $u(n)u(n+1) \neq 0$, then $\sin\gamma \cos\gamma > 0$ iff n is not a node of u and $\sin\gamma\cos\gamma < 0$ iff n is a node of u, hence (3.22) clearly holds for γ. By $\sin\Gamma\cos\gamma > 0$ we have $\sin\Gamma > 0$ iff n is not a node of u and $\sin\Gamma < 0$ iff n is a node of u, thus, (3.22) also holds for Γ.
Now, suppose we have $u(n+1) = 0$, then n is not a node of u and either $\Gamma = \pi$ or $\Gamma = 2\pi$ holds. By Lemma 3.9 we have $u(n)u(n+2) < 0$, hence

$$\sin\theta(n)\cos\theta(n+1) = (-1)^k \sin\gamma(-1)^k \cos\Gamma < 0.$$

Thus, by $\cos\Gamma < 0$, we have $\Gamma = \pi$. From $-a(n)u(n+1) = \rho(n)\cos\theta(n) = 0$ we conclude that $(-1)^k\cos\gamma = 0$, thus $\gamma = \frac{\pi}{2}$ and hence (3.22) and (3.24) hold. If $u(n) = 0$, then n is a node of u, $\gamma = \pi$, and (3.22) holds by $\sin\theta(n+1)\cos\theta(n) > 0$, i.e. $(-1)^k \sin\Gamma(-1)^k\cos\gamma > 0$. ☐

In the sequel we'll frequently use the *floor function*

$$x \mapsto \lfloor x \rfloor = \max\{n \in \mathbb{Z} \,|\, n \leqslant x\}, \tag{3.25}$$

a right-continuous step function, and the *ceiling function*

$$x \mapsto \lceil x \rceil = \min\{n \in \mathbb{Z} \,|\, n \geqslant x\}. \tag{3.26}$$

We moreover remark that $x \mapsto \lceil x \rceil - 1$ is a left-continuous analog of (3.25).

Corollary 3.11. *For all $n \in \mathbb{Z}$ we have*

$$\lceil \theta(n+1)/\pi \rceil = \begin{cases} \lceil \theta(n)/\pi \rceil + 1 & \text{if } n \text{ is a node of } u, \\ \lceil \theta(n)/\pi \rceil & \text{otherwise.} \end{cases} \tag{3.27}$$

Now we are able to count nodes of solutions of the Jacobi difference equation using Prüfer variables and the number of nodes in an interval (m, n) is given by

Theorem 3.12. *Confer Lemma 2.5 in [46]. We have*

$$\#_{(m,n)}(u) = \lceil \theta_u(n)/\pi \rceil - \lfloor \theta_u(m)/\pi \rfloor - 1. \tag{3.28}$$

Proof. We use mathematical induction: let $n = m + 1$. Then, if $u(m) = 0$, $u(n) \neq 0$ we have $\#_{(m,n)}(u) = 0$ and by Corollary 3.11

$$\lceil \theta_u(n)/\pi \rceil = \lceil \theta_u(m+1)/\pi \rceil = \lceil \underbrace{\theta_u(m)/\pi}_{\in \mathbb{Z}} \rceil + 1 = \lfloor \theta_u(m)/\pi \rfloor + 1$$

holds. If $u(m) \neq 0$ holds, then by Corollary 3.11 we have

$$\lfloor \underbrace{\theta_u(m)/\pi}_{\notin \mathbb{Z}} \rfloor = \lceil \theta_u(m)/\pi \rceil - 1 = \begin{cases} \lceil \theta_u(n)/\pi \rceil - 2 & \text{if } m \text{ is a node} \\ \lceil \theta_u(n)/\pi \rceil - 1 & \text{otherwise.} \end{cases}$$

The inductive step follows again from Corollary 3.11. □

3.3 Difference of the Prüfer angles

Again, let $u_j, j = 0, 1$, be the solutions of $\tau_j - z$ with initial values

$$u_j(n_j), u_j(n_j + 1), \quad \text{where } n_j \in \mathbb{Z}, \tag{3.29}$$

and let $\rho_j, \theta_j \in \ell(\mathbb{Z})$ be the corresponding Prüfer variables as introduced in (3.19). From now on, without loss, we assume that u_0 and u_1 correspond to the same spectral parameter z, therefore just notice that we can always replace b_1 by $b_1 - (z_1 - z_0)$. We abbreviate the difference of the Prüfer angles as

$$\Delta = \Delta_{u_0,u_1} = \theta_1 - \theta_0 \in \ell(\mathbb{Z}) \tag{3.30}$$

and adopt Lemma 3.13 and Lemma 3.14 from [4]:

Lemma 3.13. *Confer [4]. Fix some* $n \in \mathbb{Z}$*, then there exist* $k_j \in \mathbb{Z}, j = 0, 1$*, such that*

$$\begin{aligned} \theta_j(n) &= k_j\pi + \gamma_j, \qquad \gamma_j \in (0, \pi], \\ \theta_j(n+1) &= k_j\pi + \Gamma_j, \qquad \Gamma_j \in (0, 2\pi), \end{aligned} \tag{3.31}$$

where

(1) *either* u_0 *and* u_1 *have a node at* n *or both do not have a node at* n*, then*

$$\gamma_1 - \gamma_0 \in (-\frac{\pi}{2}, \frac{\pi}{2}) \quad \textit{and} \quad \Gamma_1 - \Gamma_0 \in (-\pi, \pi). \tag{3.32}$$

(2) u_1 *has no node at* n*, but* u_0 *has a node at* n*, then*

$$\gamma_1 - \gamma_0 \in (-\pi, 0) \quad \textit{and} \quad \Gamma_1 - \Gamma_0 \in (-2\pi, 0). \tag{3.33}$$

(3) u_1 *has a node at* n*, but* u_0 *has no node at* n*, then*

$$\gamma_1 - \gamma_0 \in (0, \pi) \quad \textit{and} \quad \Gamma_1 - \Gamma_0 \in (0, 2\pi). \tag{3.34}$$

Proof. Use Lemma 3.10. □

Lemma 3.14. *Confer [4]. We have*

$$\lceil \Delta(n)/\pi \rceil - 1 \le \lceil \Delta(n+1)/\pi \rceil \le \lceil \Delta(n)/\pi \rceil + 1. \tag{3.35}$$

Proof. Let $k = k_1 - k_0$, $n \in \mathbb{Z}$, then by Lemma 3.13 we have either

$$\begin{aligned} &\Delta(n) \in (k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}) &&\text{and} \quad \Delta(n+1) \in (k\pi - \pi, k\pi + \pi), \\ &\Delta(n) \in (k\pi - \pi, k\pi) &&\text{and} \quad \Delta(n+1) \in (k\pi - 2\pi, k\pi), \text{ or} \\ &\Delta(n) \in (k\pi, k\pi + \pi) &&\text{and} \quad \Delta(n+1) \in (k\pi, k\pi + 2\pi). \end{aligned}$$

In each case the lemma holds. □

Now we point out a few small lemmas which we need to relate the difference of the Prüfer angles to the nodes of the Wronskian in the next step.

Lemma 3.15. *We have*

$$W_n(u_0, u_1) = \rho_0(n)\rho_1(n) \sin \Delta(n), \tag{3.36}$$

$$W_n(u_0, u_1)u_0(n+1)u_1(n+1) = p \sin(\gamma_1 - \gamma_0) \cos \gamma_0 \cos \gamma_1, \tag{3.37}$$

$$W_{n+1}(u_0, u_1)u_0(n+1)u_1(n+1) = \tilde{p} \sin(\Gamma_1 - \Gamma_0) \cos \gamma_0 \cos \gamma_1, \tag{3.38}$$

where $p, \tilde{p} > 0$.

Proof. We have

$$\begin{aligned} W_n(u_0, u_1) &= u_0(n)a_1(n)u_1(n+1) - u_1(n)a_0(n)u_0(n+1) \\ &= \rho_0(n)\rho_1(n)(-\sin\theta_0(n)\cos\theta_1(n) + \sin\theta_1(n)\cos\theta_0(n)) \\ &= \rho_0(n)\rho_1(n)\sin(\theta_1(n) - \theta_0(n)) \\ &= \rho_0(n)\rho_1(n)(-1)^{k_1-k_0}\sin(\gamma_1(n) - \gamma_0(n)). \end{aligned}$$

The claim now holds with $p = \frac{\rho_0(n)^2\rho_1(n)^2}{a_0(n)a_1(n)}$ and $\tilde{p} = \frac{\rho_0(n)\rho_1(n)\rho_0(n+1)\rho_1(n+1)}{a_0(n)a_1(n)}$. □

Lemma 3.16. *We have*

$$\begin{aligned} u_0(n+1) = u_1(n+1) = 0 \quad &\Longrightarrow \quad W_n(u_0, u_1) = W_{n+1}(u_0, u_1) = 0, \\ u_0(n+1) = 0, u_1(n+1) \neq 0 \quad &\Longrightarrow \quad W_n(u_0, u_1)W_{n+1}(u_0, u_1) > 0, \\ u_0(n+1) \neq 0, u_1(n+1) = 0 \quad &\Longrightarrow \quad W_n(u_0, u_1)W_{n+1}(u_0, u_1) > 0. \end{aligned}$$

Proof. The first claim holds obviously. For the second claim just observe that, by Lemma 3.9,

$$\begin{aligned} &W_n(u_0, u_1)W_{n+1}(u_0, u_1) \\ &\quad = -u_0(n)u_0(n+2)a_0(n+1)a_1(n)u_1(n+1)^2 > 0 \end{aligned} \tag{3.39}$$

if $u_0(n+1) = 0, u_1(n+1) \neq 0$ and

$$\begin{aligned} &W_n(u_0, u_1)W_{n+1}(u_0, u_1) \\ &\quad = -u_1(n)u_1(n+2)a_0(n)a_1(n+1)u_0(n+1)^2 > 0 \end{aligned} \tag{3.40}$$

if $u_0(n+1) \neq 0, u_1(n+1) = 0$. □

We extract the following small corollary since we will frequently apply it in the sequel.

Corollary 3.17. *We have*

$$\left.\begin{array}{l} W_n(u_0, u_1)W_{n+1}(u_0, u_1) < 0, \ \textit{or} \\ W_n(u_0, u_1) = 0, W_{n+1}(u_0, u_1) \neq 0, \ \textit{or} \\ W_n(u_0, u_1) \neq 0, W_{n+1}(u_0, u_1) = 0 \end{array}\right\} \Longrightarrow u_0(n+1)u_1(n+1) \neq 0.$$

Moreover, $\Delta a(n) \neq 0$ *or* $\Delta b(n+1) \neq 0$ *holds.*

For the convenience of the reader we abbreviate

$$(+1) \quad \text{if} \quad \lceil \Delta(n+1)/\pi \rceil = \lceil \Delta(n)/\pi \rceil + 1,$$

$$\begin{aligned} (0) \quad &\text{if} \quad \lceil \Delta(n+1)/\pi \rceil = \lceil \Delta(n)/\pi \rceil, \text{ and} \\ (-1) \quad &\text{if} \quad \lceil \Delta(n+1)/\pi \rceil = \lceil \Delta(n)/\pi \rceil - 1. \end{aligned} \tag{3.41}$$

Now we're ready for a major step in the proof of Theorem 1.5:

Lemma 3.18. *Let $n \in \mathbb{Z}$, then*

$$\begin{aligned} (+1) \iff & W_{n+1}(u_0,u_1)u_0(n+1)u_1(n+1) > 0 \textit{ and} \\ & \textit{either} \quad W_n(u_0,u_1)W_{n+1}(u_0,u_1) < 0 \\ & \textit{or} \quad W_n(u_0,u_1) = 0, W_{n+1}(u_0,u_1) \neq 0, \end{aligned} \tag{3.42}$$

$$\begin{aligned} (-1) \iff & W_n(u_0,u_1)u_0(n+1)u_1(n+1) > 0 \textit{ and} \\ & \textit{either} \quad W_n(u_0,u_1)W_{n+1}(u_0,u_1) < 0 \\ & \textit{or} \quad W_n(u_0,u_1) \neq 0, W_{n+1}(u_0,u_1) = 0, \end{aligned} \tag{3.43}$$

$$(0) \iff \textit{otherwise.} \tag{3.44}$$

Proof. If $(+1)$, then we either have case (1) of Lemma 3.13 and $\gamma_1 - \gamma_0 \in (-\frac{\pi}{2}, 0], \Gamma_1 - \Gamma_0 \in (0,\pi)$ or we have case (3) of Lemma 3.13 and $\gamma_1 - \gamma_0 \in (0,\pi), \Gamma_1 - \Gamma_0 \in (\pi, 2\pi)$. Clearly, by (3.36), in either case we have

$$W_n(u_0,u_1)W_{n+1}(u_0,u_1) < 0 \quad \text{or} \quad W_n(u_0,u_1) = 0, W_{n+1}(u_0,u_1) \neq 0.$$

Hence, by Corollary 3.17 we have $u_0(n+1)u_1(n+1) \neq 0$, thus $\cos\gamma_0 \cos\gamma_1 \neq 0$. In case (1) of Lemma 3.13 we have $\sin(\Gamma_1 - \Gamma_0) > 0$ and $\cos\gamma_0\cos\gamma_1 > 0$ by Lemma 3.10. Hence, by (3.38) $W_{n+1}(u_0,u_1)u_0(n+1)u_1(n+1) > 0$ holds. In case (3) of Lemma 3.13 we have $\sin(\Gamma_1 - \Gamma_0) < 0$ and $\cos\gamma_0\cos\gamma_1 < 0$ by Lemma 3.10. Hence, by (3.38)

$$W_{n+1}(u_0,u_1)u_0(n+1)u_1(n+1) > 0$$

holds.

If (-1), then we either have case (1) of Lemma 3.13 and $\gamma_1 - \gamma_0 \in (0, \frac{\pi}{2}), \Gamma_1 - \Gamma_0 \in (-\pi, 0]$ or we have case (2) of Lemma 3.13 and $\gamma_1 - \gamma_0 \in (-\pi, 0), \Gamma_1 - \Gamma_0 \in (-2\pi, -\pi]$. Clearly, by (3.36), in either case we have

$$W_n(u_0,u_1)W_{n+1}(u_0,u_1) < 0 \quad \text{or} \quad W_n(u_0,u_1) \neq 0, W_{n+1}(u_0,u_1) = 0.$$

Hence, by Corollary 3.17 we have $u_0(n+1)u_1(n+1) \neq 0$, thus $\cos\gamma_0\cos\gamma_1 \neq 0$. In case (1) of Lemma 3.13 we have $\sin(\gamma_1 - \gamma_0) > 0$ and $\cos\gamma_0\cos\gamma_1 > 0$ by Lemma 3.10. Hence, by (3.37) $W_n(u_0,u_1)u_0(n+1)u_1(n+1) > 0$ holds. In case (2) of Lemma 3.13 we have $\sin(\gamma_1 - \gamma_0) < 0$ and $\cos\gamma_0\cos\gamma_1 < 0$ by Lemma 3.10. Hence, by (3.37)

$$W_n(u_0,u_1)u_0(n+1)u_1(n+1) > 0$$

holds.
On the other hand, if $W_n(u_0, u_1)W_{n+1}(u_0, u_1) < 0$ by (3.36) we have either (+1) or (−1). If, use (3.37),

$$W_n(u_0, u_1)u_0(n+1)u_1(n+1) = p\sin(\gamma_1 - \gamma_0)\cos\gamma_0\cos\gamma_1 > 0,$$

then we have either case (1) or case (2) of Lemma 3.13 and in each case we have (0) or (−1). Hence,

$$W_n(u_0, u_1)W_{n+1}(u_0, u_1) < 0 \quad \text{and} \quad W_n(u_0, u_1)u_0(n+1)u_1(n+1) > 0,$$

thus, (−1). If, use (3.37),

$$W_n(u_0, u_1)u_0(n+1)u_1(n+1) = p\sin(\gamma_1 - \gamma_0)\cos\gamma_0\cos\gamma_1 < 0,$$

then we have either case (1) or case (3) of Lemma 3.13 and in each case we have (0) or (+1). Hence,

$$W_n(u_0, u_1)W_{n+1}(u_0, u_1) < 0 \quad \text{and} \quad W_{n+1}(u_0, u_1)u_0(n+1)u_1(n+1) > 0,$$

thus (+1).
If $W_n(u_0, u_1) = 0, W_{n+1}(u_0, u_1) \neq 0$, then we have case (1) of Lemma 3.13 and $\cos\gamma_0\cos\gamma_1 > 0$ by Corollary 3.17. Thus, if $W_{n+1}(u_0, u_1)u_0(n+1)u_1(n+1) > 0$, then (3.38) implies $\sin(\Gamma_1 - \Gamma_0) > 0$, thus, (+1) holds by case (1) of Lemma 3.13. If $W_n(u_0, u_1) \neq 0, W_{n+1}(u_0, u_1) = 0$, then $\cos\gamma_0\cos\gamma_1 \neq 0$ by Corollary 3.17. If moreover $W_n(u_0, u_1)u_0(n+1)u_1(n+1) > 0$ holds, then by (3.37) $\cos\gamma_0\cos\gamma_1$ and $\sin(\gamma_1 - \gamma_0)$ are of the same sign. Hence, we have case (1) of Lemma 3.13 and (−1) or case (2) of Lemma 3.13 and (−1).
Thus, (3.42) and (3.43) hold and clearly by Lemma 3.14 we have (0) otherwise. □

Now we easily get the desired relation: from Lemma 3.18 we conclude

$$\#_n(u_0, u_1) = \lceil \Delta(n+1)/\pi \rceil - \lceil \Delta(n)/\pi \rceil, \tag{3.45}$$
$$\#_{[m,n]}(u_0, u_1) = \lceil \Delta(n)/\pi \rceil - \lceil \Delta(m)/\pi \rceil. \tag{3.46}$$

And thus obviously also

Lemma 3.19. *We have*

$$\#_{(m,n]}(u_0, u_1) = \lceil \Delta(n)/\pi \rceil - \lfloor \Delta(m)/\pi \rfloor - 1, \tag{3.47}$$
$$\#_{[m,n)}(u_0, u_1) = \lfloor \Delta(n)/\pi \rfloor - \lceil \Delta(m)/\pi \rceil + 1, \text{ and} \tag{3.48}$$
$$\#_{(m,n)}(u_0, u_1) = \lfloor \Delta(n)/\pi \rfloor - \lfloor \Delta(m)/\pi \rfloor. \tag{3.49}$$

Proof. By (3.36) we have $W_j(u_0,u_1) = 0 \iff \Delta(j)/\pi \in \mathbb{Z}$ and hence by (3.46) we have

$$\begin{aligned}
\#_{(m,n]}(u_0,u_1) &= \lceil \Delta(n)/\pi \rceil - \lceil \Delta(m)/\pi \rceil - \begin{cases} 0 & \text{if } W_m(u_0,u_1) \neq 0 \\ 1 & \text{if } W_m(u_0,u_1) = 0 \end{cases} \\
&= \lceil \Delta(n)/\pi \rceil - \lfloor \Delta(m)/\pi \rfloor - \begin{cases} 1 & \text{if } W_m(u_0,u_1) \neq 0 \\ 1 & \text{if } W_m(u_0,u_1) = 0, \end{cases} \\
\#_{[m,n)}(u_0,u_1) &= \lceil \Delta(n)/\pi \rceil - \lceil \Delta(m)/\pi \rceil + \begin{cases} 0 & \text{if } W_n(u_0,u_1) \neq 0 \\ 1 & \text{if } W_n(u_0,u_1) = 0 \end{cases} \\
&= \lfloor \Delta(n)/\pi \rfloor - \lceil \Delta(m)/\pi \rceil + 1, \\
\#_{(m,n)}(u_0,u_1) &= \lceil \Delta(n)/\pi \rceil - \lceil \Delta(m)/\pi \rceil \\
&\quad + \begin{cases} 0 & \text{if } W_n(u_0,u_1) \neq 0 \\ 1 & \text{if } W_n(u_0,u_1) = 0 \end{cases} - \begin{cases} 0 & \text{if } W_m(u_0,u_1) \neq 0 \\ 1 & \text{if } W_m(u_0,u_1) = 0 \end{cases} \\
&= \lfloor \Delta(n)/\pi \rfloor + 1 - (\lfloor \Delta(m)/\pi \rfloor + 1).
\end{aligned}$$

□

Lemma 3.20. *We have*

$$\#_{[m,n]}(u_0,u_1) = -\#_{(m,n)}(u_1,u_0), \tag{3.50}$$
$$\#_{(m,n]}(u_0,u_1) = -\#_{[m,n)}(u_1,u_0). \tag{3.51}$$

If $W_m(u_0,u_1) \neq 0$ and $W_n(u_0,u_1) \neq 0$, then

$$\#_{[m,n]}(u_0,u_1) = -\#_{[m,n]}(u_1,u_0). \tag{3.52}$$

Proof. By $\lceil x \rceil = -\lfloor -x \rfloor$ we have

$$\begin{aligned}
\#_{[m,n]}(u_0,u_1) &= \lceil (\theta_1(n) - \theta_0(n))/\pi \rceil - \lceil (\theta_1(m) - \theta_0(m))/\pi \rceil \\
&= -(\lfloor (\theta_0(n) - \theta_1(n))/\pi \rfloor - \lfloor (\theta_0(m) - \theta_1(m))/\pi \rfloor) \\
&= -\#_{(m,n)}(u_1,u_0)
\end{aligned}$$

and

$$\begin{aligned}
\#_{(m,n]}(u_0,u_1) &= \lceil (\theta_1(n) - \theta_0(n))/\pi \rceil - \lfloor (\theta_1(m) - \theta_0(m))/\pi \rfloor - 1 \\
&= -(\lfloor (\theta_0(n) - \theta_1(n))/\pi \rfloor - \lceil (\theta_0(m) - \theta_1(m))/\pi \rceil + 1) \\
&= -\#_{[m,n)}(u_1,u_0).
\end{aligned}$$

□

If $\Delta a = 0$ holds, then (1.25) reduces to (1.10), which is also (1.8) from [4], see the following

Lemma 3.21. *Let (1.25) and $a_0 = a_1$, then (1.10) holds, which is*

$$\#_n(u_0, u_1) = \begin{cases} 1 & \text{if } b_0(n+1) - z_0 - b_1(n+1) + z_1 > 0 \text{ and} \\ & \quad \text{either } W_n(u_0,u_1)W_{n+1}(u_0,u_1) < 0 \\ & \quad \text{or } W_n(u_0,u_1) = 0 \text{ and } W_{n+1}(u_0,u_1) \neq 0 \\ -1 & \text{if } b_0(n+1) - z_0 - b_1(n+1) + z_1 < 0 \text{ and} \\ & \quad \text{either } W_n(u_0,u_1)W_{n+1}(u_0,u_1) < 0 \\ & \quad \text{or } W_n(u_0,u_1) \neq 0 \text{ and } W_{n+1}(u_0,u_1) = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3.53}$$

Proof. Without loss, let $z_0 = z_1$. If we have $W_n(u_0, u_1)W_{n+1}(u_0, u_1) > 0$ or $W_n(u_0, u_1) = W_{n+1}(u_0, u_1) = 0$, then the claim holds obviously. Otherwise, by (3.5) and Corollary 3.17 we have

$$\begin{aligned} & W_{n+1}(u_0,u_1)u_0(n+1)u_1(n+1) - W_n(u_0,u_1)u_0(n+1)u_1(n+1) \\ & \quad = \Delta b(n+1)u_0(n+1)^2u_1(n+1)^2 \neq 0. \end{aligned} \tag{3.54}$$

If $W_n(u_0, u_1) = 0, W_{n+1}(u_0, u_1) \neq 0$ holds, then $W_{n+1}(u_0, u_1)u_0(n+1)u_1(n+1)$ and $\Delta b(n+1)$ are of the same sign by (3.54).
If $W_n(u_0, u_1) \neq 0, W_{n+1}(u_0, u_1) = 0$, then $W_n(u_0, u_1)u_0(n+1)u_1(n+1)$ and $\Delta b(n+1) \neq 0$ are of opposite sign by (3.54).
Now, suppose $W_n(u_0, u_1)W_{n+1}(u_0, u_1) < 0$ holds: if $\#(u_0, u_1) = 1$, then

$$W_{n+1}(u_0,u_1)u_0(n+1)u_1(n+1) > 0, W_n(u_0,u_1)u_0(n+1)u_1(n+1) < 0,$$

thus by (3.54) we have $\Delta b(n+1) > 0$. If $\#(u_0, u_1) = -1$, then

$$W_n(u_0,u_1)u_0(n+1)u_1(n+1) > 0, W_{n+1}(u_0,u_1)u_0(n+1)u_1(n+1) < 0,$$

and hence $\Delta b(n+1) < 0$ holds by (3.54). □

Remark 3.22. *Consider (1.25), then*

$$\begin{aligned} & W_n(u_0,u_1)W_{n+1}(u_0,u_1) \neq 0 \quad \text{or} \quad W_n(u_0,u_1) = W_{n+1}(u_0,u_1) = 0 \\ & \quad \Longrightarrow \#_n(u_0,u_1) = -\#_n(u_1,u_0), \end{aligned} \tag{3.55}$$

$$W_n(u_0,u_1)W_{n+1}(u_0,u_1) < 0 \implies \#_n(u_0,u_1) \neq 0 \tag{3.56}$$

by Corollary 3.17. Moreover, if $W_n(u_0, u_1) = 0$ and $W_{n+1}(u_0, u_1) \neq 0$ holds, then $u_0(n) = 0 \iff u_1(n) = 0$.

Chapter 4

Finite Jacobi matrices

Now we're prepared to prove Theorem 1.5. Therefore we normalize the solutions of τ fulfilling the left/right Dirichlet boundary condition of the Jacobi matrix J from (1.8) so that

$$s_-(0) = 0, s_-(1) = 1, \qquad s_+(N) = 0, s_+(N+1) = 1. \tag{4.1}$$

Fix a base point $n_0 = N$ or $n_0 = 0$, then by $s_\pm(n_0) = 0$ we have $\sin\theta_\pm(n_0) = 0$ and by $s_\pm(n_0+1) = 1$ we have $-a(n_0)s_\pm(n_0+1) = \rho_s(n_0)\cos\theta_\pm(n_0) > 0$, hence by $\theta_\pm(n_0) \in (-\pi, \pi]$ we have

$$\theta_\pm(n_0) = 0. \tag{4.2}$$

From Theorem 3.12 we obtain the following

Corollary 4.1. *We find*

$$\#_{(0,N)}(s_-) = \lceil \theta_{s_-}(N)/\pi \rceil - 1, \tag{4.3}$$

$$\#_{(0,N)}(s_+) = -\lfloor \theta_{s_+}(0)/\pi \rfloor - 1. \tag{4.4}$$

Recall a few well-known findings about the spectrum of Jacobi matrices:

Lemma 4.2. *Confer [42]. We have*

$$z \in \sigma(J) \iff s_-(z, N) = 0 \iff s_+(z, 0) = 0. \tag{4.5}$$

Lemma 4.3. *Confer [42]. The Jacobi matrix J has $N-1$ real and simple eigenvalues.*

Proof. Since J is Hermitian all eigenvalues are real: let $z \in \sigma(J), Jv = zv$ and $\|v\| = 1$, then

$$z = \langle v, zv\rangle = \langle v, Jv\rangle = \langle Jv, v\rangle = \overline{z}.$$

It can easily be seen that every eigenvector u corresponding to z fulfills $\tau u = zu$

and $u(0) = 0$. Hence, by $W_0(s_-(z), u) = 0$, the solutions $s_-(z)$ and u are linearly dependent by Lemma 2.24. □

Theorem 4.4. *Confer [46] or Theorem 4.7 in [42]. We have*

$$E_{(-\infty,\lambda)}(J) = \#_{(0,N)}(s_-(\lambda)) \tag{4.6}$$
$$= \#_{(0,N)}(s_+(\lambda)), \tag{4.7}$$

where $E_\Omega(J)$ denotes the number of eigenvalues of J in $\Omega \subseteq \mathbb{R}$.

Now we can already relate the spectrum to the Prüfer transformation.

Lemma 4.5. *Let $a_0, a_1 < 0$, then*

$$\begin{aligned} &E_{(-\infty,\lambda_1)}(J_1) - E_{(-\infty,\lambda_0)}(J_0) \\ &\quad = \lceil \Delta_{s_{0,+}(\lambda_0),s_{1,-}(\lambda_1)}(N)/\pi \rceil - \lceil \Delta_{s_{0,+}(\lambda_0),s_{1,-}(\lambda_1)}(0)/\pi \rceil \\ &\quad = \lfloor \Delta_{s_{0,-}(\lambda_0),s_{1,+}(\lambda_1)}(N)/\pi \rfloor - \lfloor \Delta_{s_{0,-}(\lambda_0),s_{1,+}(\lambda_1)}(0)/\pi \rfloor, \end{aligned} \tag{4.8}$$

$$\begin{aligned} &E_{(-\infty,\lambda_1)}(J_1) - E_{(-\infty,\lambda_0]}(J_0) \\ &\quad = \lceil \Delta_{s_{0,\pm}(\lambda_0),s_{1,\mp}(\lambda_1)}(N)/\pi \rceil - \lfloor \Delta_{s_{0,\pm}(\lambda_0),s_{1,\mp}(\lambda_1)}(0)/\pi \rfloor - 1, \end{aligned} \tag{4.9}$$

$$\begin{aligned} &E_{(-\infty,\lambda_1]}(J_1) - E_{(-\infty,\lambda_0)}(J_0) \\ &\quad = \lfloor \Delta_{s_{0,\pm}(\lambda_0),s_{1,\mp}(\lambda_1)}(N)/\pi \rfloor - \lceil \Delta_{s_{0,\pm}(\lambda_0),s_{1,\mp}(\lambda_1)}(0)/\pi \rceil + 1, \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} &E_{(-\infty,\lambda_1]}(J_1) - E_{(-\infty,\lambda_0]}(J_0) \\ &\quad = \lceil \Delta_{s_{0,-}(\lambda_0),s_{1,+}(\lambda_1)}(N)/\pi \rceil - \lceil \Delta_{s_{0,-}(\lambda_0),s_{1,+}(\lambda_1)}(0)/\pi \rceil \\ &\quad = \lfloor \Delta_{s_{0,+}(\lambda_0),s_{1,-}(\lambda_1)}(N)/\pi \rfloor - \lfloor \Delta_{s_{0,+}(\lambda_0),s_{1,-}(\lambda_1)}(0)/\pi \rfloor, \end{aligned} \tag{4.11}$$

where $\Delta_{u,v} = \theta_v - \theta_u \in \ell(\mathbb{Z})$ and $s_\pm$ are the solutions from (4.1).

Proof. We abbreviate $s_{j,\pm} = s_{j,\pm}(\lambda_j)$, then by Theorem 4.4, Corollary 4.1, and $-\lceil x \rceil = \lfloor -x \rfloor$ for all $x \in \mathbb{R}$ we have

$$\begin{aligned} &E_{(-\infty,\lambda_1)}(J_1) - E_{(-\infty,\lambda_0)}(J_0) \\ &\quad = \#_{(0,N)}(s_{1,-}) - \#_{(0,N)}(s_{0,+}) \\ &\quad = \lceil \theta_{s_{1,-}}(N)/\pi \rceil + \lfloor \theta_{s_{0,+}}(0)/\pi \rfloor = \lceil \theta_{s_{1,-}}(N)/\pi \rceil - \lceil -\theta_{s_{0,+}}(0)/\pi \rceil \\ &\quad = \lceil (\theta_{s_{1,-}}(N) - \theta_{s_{0,+}}(N))/\pi \rceil - \lceil (\theta_{s_{1,-}}(0) - \theta_{s_{0,+}}(0))/\pi \rceil \\ &\quad = \lceil \Delta_{s_{0,+},s_{1,-}}(N)/\pi \rceil - \lceil \Delta_{s_{0,+},s_{1,-}}(0)/\pi \rceil \\ &= -(E_{(-\infty,\lambda_0)}(J_0) - E_{(-\infty,\lambda_1)}(J_1)) \\ &\quad = -(\lceil \Delta_{s_{1,+},s_{0,-}}(N)/\pi \rceil - \lceil \Delta_{s_{1,+},s_{0,-}}(0)/\pi \rceil) \\ &\quad = \lfloor \Delta_{s_{0,-},s_{1,+}}(N)/\pi \rfloor - \lfloor \Delta_{s_{0,-},s_{1,+}}(0)/\pi \rfloor. \end{aligned}$$

By Lemma 4.2 we have

$$\begin{aligned}\lambda_0 \in \sigma(J_0) &\iff \Delta_{s_{0,-},s_{1,+}}(N)/\pi \in \mathbb{Z} \iff \Delta_{s_{0,+},s_{1,-}}(0)/\pi \in \mathbb{Z}, \\ \lambda_1 \in \sigma(J_1) &\iff \Delta_{s_{0,+},s_{1,-}}(N)/\pi \in \mathbb{Z} \iff \Delta_{s_{0,-},s_{1,+}}(0)/\pi \in \mathbb{Z}\end{aligned} \tag{4.12}$$

and hence,

$$\begin{aligned}&E_{(-\infty,\lambda_1)}(J_1) - E_{(-\infty,\lambda_0)}(J_0) \\ &\quad = \lceil \Delta_{s_{0,\pm},s_{1,\mp}}(N)/\pi \rceil - \lfloor \Delta_{s_{0,\pm},s_{1,\mp}}(0)/\pi \rfloor - \begin{cases} 1 & \text{if } \lambda_0 \notin \sigma(J_0) \\ 0 & \text{if } \lambda_0 \in \sigma(J_0) \end{cases} \\ &\quad = \lfloor \Delta_{s_{0,\pm},s_{1,\mp}}(N)/\pi \rfloor - \lceil \Delta_{s_{0,\pm},s_{1,\mp}}(0)/\pi \rceil + \begin{cases} 1 & \text{if } \lambda_1 \notin \sigma(J_1) \\ 0 & \text{if } \lambda_1 \in \sigma(J_1). \end{cases}\end{aligned} \tag{4.13}$$

By (4.13) we now have

$$\begin{aligned}&E_{(-\infty,\lambda_1)}(J_1) - E_{(-\infty,\lambda_0]}(J_0) \\ &\quad = \lceil \Delta_{s_{0,\pm},s_{1,\mp}}(N)/\pi \rceil - \lfloor \Delta_{s_{0,\pm},s_{1,\mp}}(0)/\pi \rfloor - \begin{cases} 1 & \text{if } \lambda_0 \notin \sigma(J_0) \\ 1 & \text{if } \lambda_0 \in \sigma(J_0), \end{cases} \\ &E_{(-\infty,\lambda_1]}(J_1) - E_{(-\infty,\lambda_0)}(J_0) \\ &\quad = \lfloor \Delta_{s_{0,\pm},s_{1,\mp}}(N)/\pi \rfloor - \lceil \Delta_{s_{0,\pm},s_{1,\mp}}(0)/\pi \rceil + \begin{cases} 1 & \text{if } \lambda_1 \notin \sigma(J_1) \\ 1 & \text{if } \lambda_1 \in \sigma(J_1), \end{cases}\end{aligned} \tag{4.14}$$

and by (4.14) we have

$$\begin{aligned}&E_{(-\infty,\lambda_1]}(J_1) - E_{(-\infty,\lambda_0]}(J_0) \\ &\quad = \lfloor \Delta_{s_{0,\mp},s_{1,\pm}}(N)/\pi \rfloor - \lceil \Delta_{s_{0,\mp},s_{1,\pm}}(0)/\pi \rceil + 1 - \begin{cases} 0 & \text{if } \lambda_0 \notin \sigma(J_0) \\ 1 & \text{if } \lambda_0 \in \sigma(J_0). \end{cases}\end{aligned}$$

The last claim now follows from (4.12). □

And finally we obtain Theorem 1.5 except that we count one possible node too much.

Theorem 4.6. *Let $a_0, a_1 < 0$. Then,*

$$\begin{aligned}&E_{(-\infty,z_1)}(J_1) - E_{(-\infty,z_0]}(J_0) \\ &\quad = \#_{(0,N]}(u_{0,+}(z_0), u_{1,-}(z_1)) = \#_{(0,N]}(u_{0,-}(z_0), u_{1,+}(z_1)),\end{aligned} \tag{4.15}$$

$$\begin{aligned}&E_{(-\infty,z_1)}(J_1) - E_{(-\infty,z_0)}(J_0) \\ &\quad = \#_{[0,N]}(u_{0,+}(z_0), u_{1,-}(z_1)) = \#_{(0,N)}(u_{0,-}(z_0), u_{1,+}(z_1)),\end{aligned} \tag{4.16}$$

$$E_{(-\infty,z_1]}(J_1) - E_{(-\infty,z_0]}(J_0)$$
$$= \#_{(0,N)}(u_{0,+}(z_0), u_{1,-}(z_1)) = \#_{[0,N]}(u_{0,-}(z_0), u_{1,+}(z_1)), \tag{4.17}$$
$$E_{(-\infty,z_1]}(J_1) - E_{(-\infty,z_0)}(J_0)$$
$$= \#_{[0,N)}(u_{0,+}(z_0), u_{1,-}(z_1)) = \#_{[0,N)}(u_{0,-}(z_0), u_{1,+}(z_1)), \tag{4.18}$$

where $E_\Omega(J_j), j = 0, 1$, is the number of eigenvalues of J_j in $\Omega \subseteq \mathbb{R}$ and $u_{j,\pm}(z_j)$ are solutions fulfilling the right/left Dirichlet boundary condition of J, i.e. $u_{j,+}(z_j, N) = u_{j,-}(z_j, 0) = 0$.

Proof. By Lemma 4.5 we have

$$\begin{aligned}
&E_{(-\infty,\lambda_1)}(J_1) - E_{(-\infty,\lambda_0)}(J_0)\\
&\quad = \lceil \Delta_{s_{0,+}(\lambda_0), s_{1,-}(\lambda_1)}(N)/\pi \rceil - \lceil \Delta_{s_{0,+}(\lambda_0), s_{1,-}(\lambda_1)}(0)/\pi \rceil\\
&\quad = \lfloor \Delta_{s_{0,-}(\lambda_0), s_{1,+}(\lambda_1)}(N)/\pi \rfloor - \lfloor \Delta_{s_{0,-}(\lambda_0), s_{1,+}(\lambda_1)}(0)/\pi \rfloor,\\
&E_{(-\infty,\lambda_1]}(J_1) - E_{(-\infty,\lambda_0]}(J_0)\\
&\quad = \lceil \Delta_{s_{0,-}(\lambda_0), s_{1,+}(\lambda_1)}(N)/\pi \rceil - \lceil \Delta_{s_{0,-}(\lambda_0), s_{1,+}(\lambda_1)}(0)/\pi \rceil\\
&\quad = \lfloor \Delta_{s_{0,+}(\lambda_0), s_{1,-}(\lambda_1)}(N)/\pi \rfloor - \lfloor \Delta_{s_{0,+}(\lambda_0), s_{1,-}(\lambda_1)}(0)/\pi \rfloor,\\
&E_{(-\infty,\lambda_1)}(J_1) - E_{(-\infty,\lambda_0]}(J_0)\\
&\quad = \lceil \Delta_{s_{0,\pm}(\lambda_0), s_{1,\mp}(\lambda_1)}(N)/\pi \rceil - \lfloor \Delta_{s_{0,\pm}(\lambda_0), s_{1,\mp}(\lambda_1)}(0)/\pi \rfloor - 1,\\
&E_{(-\infty,\lambda_1]}(J_1) - E_{(-\infty,\lambda_0)}(J_0)\\
&\quad = \lfloor \Delta_{s_{0,\pm}(\lambda_0), s_{1,\mp}(\lambda_1)}(N)/\pi \rfloor - \lceil \Delta_{s_{0,\pm}(\lambda_0), s_{1,\mp}(\lambda_1)}(0)/\pi \rceil + 1,
\end{aligned}$$

now use Lemma 3.19 and (3.46). Moreover, we can replace $s_\pm$ by a constant multiple $u_\pm$ because they have equally many nodes. □

For the finite case everything that remains to be shown now is that under certain assumptions there's indeed no node at the place $N - 1$.

Proof of Theorem 1.5. The solutions u_- and u_+ in Theorem 4.6 depend on the coefficients $a(0)$ and $a(N-1)$ of τ, although J (and hence also $\sigma(J)$) doesn't depend on them. We choose

$$a_0(N-1) = a_1(N-1). \tag{4.19}$$

Then, by $\Delta a(N-1) = 0$ and (3.5) we have

$$\begin{aligned}
&W_N(u_{0,+}(z_0)), u_{1,-}(z_1)) - W_{N-1}(u_{0,+}(z_0), u_{1,-}(z_1)) \qquad (4.20)\\
&\quad = (b_0(N) - z_0 - b_1(N) + z_1) u_{0,+}(z_0, N) u_{1,-}(z_1, N) = 0.
\end{aligned}$$

Hence there's no node at $N-1$. The same holds for $W(u_{0,-}(z_0), u_{1,+}(z_1))$, thus

$\#_{N-1}(u_{0,\pm}(z_0), u_{1,\mp}(z_1)) = 0$ and

$$W_{N-1}(u_{0,\pm}(z_0), u_{1,\mp}(z_1)) = W_N(u_{0,\pm}(z_0), u_{1,\mp}(z_1)). \tag{4.21}$$

□

Moreover, we finally note a few additional properties of the nodes of the Wronskian.

Corollary 4.7. *We have*

$$\#_{[0,N]}(u_{0,\pm}(\lambda), u_{1,\mp}(\lambda)) = -\#_{[0,N]}(u_{1,\pm}(\lambda), u_{0,\mp}(\lambda)), \tag{4.22}$$

where $u_{j,\pm}(\lambda)$ *denotes a solution fulfilling the right/left Dirichlet boundary condition of* J_j*, where* $j = 0, 1$*.*

Remark 4.8. *We have*

$$\begin{aligned}
&\#_{[0,N]}(u_{0,+}(\lambda), u_{3,-}(\lambda)) \\
&\quad = \#_{[0,N)}(u_{0,+}(\lambda), u_{1,-}(\lambda)) + \#_{[0,N]}(u_{1,-}(\lambda), u_{2,+}(\lambda)) \qquad (4.23)\\
&\qquad + \#_{(0,N]}(u_{2,+}(\lambda), u_{3,-}(\lambda)),\\
&\#_{[0,N]}(u_{0,-}(\lambda), u_{3,+}(\lambda)) \\
&\quad = \#_{(0,N]}(u_{0,-}(\lambda), u_{1,+}(\lambda)) + \#_{[0,N]}(u_{1,+}(\lambda), u_{2,-}(\lambda)) \qquad (4.24)\\
&\qquad + \#_{[0,N)}(u_{2,-}(\lambda), u_{3,+}(\lambda)),
\end{aligned}$$

where $u_{j,\pm}(\lambda)$ *denotes a solution fulfilling the right/left Dirichlet boundary condition of* J_j*, where* $j = 0, 1$*.*

Proof. Abbreviate $u = u(\lambda)$, then by Theorem 1.5 we have

$$\begin{aligned}
&\#_{[0,N]}(u_{0,+}, u_{3,-}) = E_{(-\infty,\lambda)}(J_3) - E_{(-\infty,\lambda)}(J_0)\\
&\quad = E_{(-\infty,\lambda]}(J_1) - E_{(-\infty,\lambda)}(J_0) + E_{(-\infty,\lambda]}(J_2) - E_{(-\infty,\lambda]}(J_1)\\
&\qquad + E_{(-\infty,\lambda)}(J_3) - E_{(-\infty,\lambda]}(J_2)\\
&\quad = \#_{[0,N)}(u_{0,+}, u_{1,-}) + \#_{[0,N]}(u_{1,-}, u_{2,+}) + \#_{(0,N]}(u_{2,+}, u_{3,-}),\\
&\#_{[0,N]}(u_{0,-}, u_{3,+}) = E_{(-\infty,\lambda]}(J_3) - E_{(-\infty,\lambda]}(J_0)\\
&\quad = E_{(-\infty,\lambda)}(J_1) - E_{(-\infty,\lambda]}(J_0) + E_{(-\infty,\lambda)}(J_2) - E_{(-\infty,\lambda)}(J_1)\\
&\qquad + E_{(-\infty,\lambda]}(J_3) - E_{(-\infty,\lambda)}(J_2)\\
&\quad = \#_{(0,N]}(u_{0,-}, u_{1,+}) + \#_{[0,N]}(u_{1,+}, u_{2,-}) + \#_{[0,N)}(u_{2,-}, u_{3,+}).
\end{aligned}$$

□

Chapter 5

Determinants

We'll drop our main assumption $a(n) < 0$ for the rest of this chapter and consider Jacobi matrices

$$J = \begin{pmatrix} b(1) & a(1) & & \\ a(1) & \ddots & \ddots & \\ & \ddots & & a(N-2) \\ & & a(N-2) & b(N-1) \end{pmatrix} \tag{5.1}$$

from (1.8), where just

$$a(n) \neq 0 \tag{5.2}$$

holds for all n. We denote the determinants of the top left submatrices of J by

$$m_-(n) = \begin{vmatrix} b(1) & a(1) & & \\ a(1) & \ddots & \ddots & \\ & \ddots & & a(n-1) \\ & & a(n-1) & b(n) \end{vmatrix} \tag{5.3}$$

and the determinants of the bottom right submatrices of J by

$$m_+(n) = \begin{vmatrix} b(n) & a(n) & & \\ a(n) & \ddots & \ddots & \\ & \ddots & & a(N-2) \\ & & a(N-2) & b(N-1) \end{vmatrix}, \tag{5.4}$$

where $n = 1, \dots, N-1$. To simplify notation we set

$$m_-(0) = m_+(N) = 1, \tag{5.5}$$

$$m_-(-1) = m_+(N+1) = 0,$$

and, without loss, $a(0) = a(N-1) = -1$.

5.1 Solutions and leading principal minors

For the rest of this chapter let ψ_- and ψ_+ be solutions of $\tau\psi = 0$ fulfilling the left/right Dirichlet boundary condition of J. We normalize the solutions such that

$$\begin{aligned} &\psi_-(0) = 0, \quad \psi_-(1) = 1 = m_-(0), \\ &\psi_+(N-1) = 1 = m_+(N), \quad \psi_+(N) = 0. \end{aligned} \tag{5.6}$$

Now, we find

Lemma 5.1. *Let $a(n) \neq 0$ for all n, then*

$$\psi_-(n) = \frac{m_-(n-1)}{\prod_{j=1}^{n-1} -a(j)}, \tag{5.7}$$

$$\psi_+(n) = \frac{m_+(n+1)}{\prod_{i=n}^{N-2} -a(i)} \tag{5.8}$$

for all $n = 0, \dots, N$. If $a < 0$, then $m_-(n-1)$ and $\psi_-(n)$ as well as $m_+(n+1)$ and $\psi_+(n)$ are of the same sign. Obviously, ψ can be replaced by a solution of $(\tau - z)\psi = 0$ if J is replaced by $J - z$.

Proof. For the first claim look at $\psi_-(1) = 1 = m_-(0)$ and

$$\psi_-(2) = -a(1)^{-1}(b(1)\psi_-(1) + a(0)\psi_-(0)) = -a(1)^{-1}m_-(1).$$

For $n \geqslant 2$ by the Laplace expansion we have

$$m_-(n) = b(n)m_-(n-1) - a(n-1)^2 m_-(n-2) \tag{5.9}$$

and hence

$$\begin{aligned} \frac{-m_-(n)}{\prod_{j=1}^{n-1} -a(j)} &= -a(n-1)\frac{m_-(n-2)}{\prod_{j=1}^{n-2} -a(j)} - b(n)\frac{m_-(n-1)}{\prod_{j=1}^{n-1} -a(j)} \\ &= -a(n-1)\psi_-(n-1) - b(n)\psi_-(n) = a(n)\psi_-(n+1). \end{aligned}$$

For the second claim look at $\psi_+(N-1) = m_+(N) = 1$ and

$$\psi_+(N-2) = \frac{b(N-1)\psi_+(N-1)}{-a(N-2)} = \frac{m_+(N-1)}{-a(N-2)}$$

by $\psi_+(N) = 0$. For the inductive step we have

$$\begin{aligned}\psi_+(n-1) &= \frac{b(n)\psi_+(n) + a(n)\psi_+(n+1)}{-a(n-1)} \\ &= \frac{b(n)m_+(n+1) - a(n)^2 m_+(n+2)}{\prod_{i=n-1}^{N-2} -a(i)} = \frac{m_+(n)}{\prod_{i=n-1}^{N-2} -a(i)},\end{aligned}$$

which holds by $m_+(n) = b(n)m_+(n+1) - a(n)^2 m_+(n+2)$. □

Formula (5.7) can be found in II.1.(8) from [18], confer also [35] and (5.28) which is equation (1.65) in [42].

Lemma 5.2. *Let $J > 0$, then*

$$m_-(N-2) > \frac{\prod_{n=1}^{N-2} a(n)^2}{\prod_{n=2}^{N-1} b(n)} > 0 \tag{5.10}$$

and

$$\psi_-(N-1) > \frac{\prod_{n=1}^{N-2} -a(n)}{\prod_{n=2}^{N-1} b(n)} > 0. \tag{5.11}$$

Proof. By the Laplace expansion and Sylvester's criterion we have

$$m_-(n) = b(n)m_-(n-1) - a(n-1)^2 m_-(n-2) > 0$$

for all $n = 1, \dots, N-1$, thus $b(n) > a(n-1)^2 \frac{m_-(n-2)}{m_-(n-1)} > 0$ and

$$\prod_{n=2}^{N-1} b(n) > \frac{m_-(0)\dots m_-(N-3)}{m_-(1)\dots m_-(N-2)} \prod_{n=2}^{N-1} a(n-1)^2 > 0.$$

Now, use $\psi_-(N-1)\prod_{j=1}^{N-2} -a(j) = m_-(N-2)$ from (5.7). □

5.2 A Wronskian of determinants

In this section we demonstrate how Theorem 1.5 can be translated to subdeterminants of J_0 and J_1, therefore we assume

$$a = a_0 = a_1. \tag{5.12}$$

Lemma 5.3. *We find*

$$W_n(\psi_{0,-}, \psi_{1,+}) \prod_{i=1}^{N-2} -a(i) = \begin{vmatrix} m_{0,-}(n) & a(n)m_{1,+}(n+2) \\ a(n)m_{0,-}(n-1) & m_{1,+}(n+1) \end{vmatrix} = \Phi_n$$

for all $n = 0, \dots, N-1$. *Moreover,*

$$\Phi_0 = \det J_1 \quad \text{and} \quad \Phi_{N-1} = \det J_0. \tag{5.13}$$

Proof. For all $n = 1, \dots, N-2$ we have

$$\begin{aligned}
&- a(n)^{-1} W_n(\psi_{0,-}, \psi_{1,+}) \\
&\quad = \psi_{1,+}(n)\psi_{0,-}(n+1) - \psi_{0,-}(n)\psi_{1,+}(n+1) \\
&\quad = \frac{m_{1,+}(n+1)m_{0,-}(n)}{-a(n)\prod_{i=1}^{N-2} -a(i)} - \frac{-a(n)m_{0,-}(n-1)m_{1,+}(n+2)}{\prod_{i=1}^{N-2} -a(i)}
\end{aligned}$$

by Lemma 5.1. Hence,

$$\begin{aligned}
&- a(n)^{-1} W_n(\psi_{0,-}, \psi_{1,+}) \prod_{i=1}^{N-2} -a(i) \\
&\quad = a(n)m_{0,-}(n-1)m_{1,+}(n+2) - a(n)^{-1}m_{1,+}(n+1)m_{0,-}(n).
\end{aligned}$$

Moreover,

$$- a(0)^{-1} W_0(\psi_{0,-}, \psi_{1,+}) = \psi_{1,+}(0) = \det J_1 \prod_{i=0}^{N-2} -a(i)^{-1},$$

where $a(0) = -1$, and

$$- a(N-1)^{-1} W_{N-1}(\psi_{0,-}, \psi_{1,+}) = \psi_{0,-}(N) = \det J_0 \prod_{j=1}^{N-2} -a(j)^{-1}.$$

□

Now, we weight the nodes of Φ in the same way as we weight nodes of the Wronskian of solutions, that is,

$$\#_n\Phi = \begin{cases} 1 & \text{if } b_0(n+1) - b_1(n+1) > 0 \text{ and} \\ & \quad \text{either } \Phi_n\Phi_{n+1} < 0 \\ & \quad \text{or } \Phi_n = 0 \text{ and } \Phi_{n+1} \neq 0 \\ -1 & \text{if } b_0(n+1) - b_1(n+1) < 0 \text{ and} \\ & \quad \text{either } \Phi_n\Phi_{n+1} < 0 \\ & \quad \text{or } \Phi_n \neq 0 \text{ and } \Phi_{n+1} = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{5.14}$$

With this definition we find

Theorem 5.4. *Let $a < 0$, then*

$$E_{(-\infty,0]}(J_1) - E_{(-\infty,0]}(J_0) = \sum_{j=0}^{N-2} \#_j \Phi. \tag{5.15}$$

Proof. Obviously, by $a < 0$ and Lemma 5.3 the sequences $W(\psi_{0,-}, \psi_{1,+})$ and Φ are of the same sign for all $n = 0, \dots, N-1$, and thus

$$\#_j \Phi = \#_j(\psi_{0,-}, \psi_{1,+}) \tag{5.16}$$

for all $j = 0, \dots, N-2$. Further, by Theorem 1.5

$$\begin{aligned} &E_{(-\infty,0]}(J_1) - E_{(-\infty,0]}(J_0) \\ &\quad = \#_{[0,N-1]}(\psi_{0,-}, \psi_{1,+}) = \sum_{j=0}^{N-2} \#_j(\psi_{0,-}, \psi_{1,+}) = \sum_{j=0}^{N-2} \#_j \Phi \end{aligned}$$

holds. □

Of course, the more general case (different a's) analogously translates to the principal minors. And further we easily obtain the 'derivative' of Φ:

Theorem 5.5. *Let $a < 0$, then*

$$\Phi_n - \Phi_{n-1} = (b_0(n) - b_1(n)) m_{0,-}(n-1) m_{1,+}(n+1) \tag{5.17}$$

holds for all $n = 1, \dots, N-1$.

Proof. By Lemma 5.3, (3.5), and Lemma 5.1 we have

$$\begin{aligned} \Phi_n - \Phi_{n-1} &= W_n(\psi_{0,-}, \psi_{1,+}) \prod_{i=1}^{N-2} -a(i) - W_{n-1}(\psi_{0,-}, \psi_{1,+}) \prod_{i=1}^{N-2} -a(i) \\ &= (b_0(n) - b_1(n)) \psi_{0,-}(n) \psi_{1,+}(n) \prod_{i=1}^{N-2} -a(i) \\ &= (b_0(n) - b_1(n)) m_{0,-}(n-1) m_{1,+}(n+1). \end{aligned}$$

□

And obviously, we find analogous theorems if we consider $m_{0,+}$ and $m_{1,-}$:

Lemma 5.6. *For all $n = 0, \dots, N-1$ we have*

$$\begin{aligned} &W_n(\psi_{0,+}, \psi_{1,-}) \prod_{i=1}^{N-2} -a(i) \\ &= \begin{vmatrix} a(n) m_{0,+}(n+2) & m_{1,-}(n) \\ m_{0,+}(n+1) & a(n) m_{1,-}(n-1) \end{vmatrix} = \tilde{\Phi}_n. \end{aligned} \tag{5.18}$$

Moreover,

$$\tilde{\Phi}_0 = -\det J_0 \quad \textit{and} \quad \tilde{\Phi}_{N-1} = -\det J_1. \tag{5.19}$$

Proof. Use Lemma 5.3 and $W_n(\psi_{0,+}, \psi_{1,-}) = -W_n(\psi_{1,-}, \psi_{0,+})$. □

Theorem 5.7. *Let $a = a_0 = a_1 < 0$, then*

$$E_{(-\infty,0)}(J_1) - E_{(-\infty,0)}(J_0) = \sum_{j=0}^{N-2} \#_j \tilde{\Phi}. \tag{5.20}$$

Proof. Obviously, by $a < 0$ and Lemma 5.3 the sequences $W_n(\psi_{0,+}, \psi_{1,-})$ and $\tilde{\Phi}_n$ are of the same sign for all $n = 0, \dots, N-1$ and thus $\#_j(\psi_{0,+}, \psi_{1,-}) = \#_j \tilde{\Phi}$ for all $j = 0, \dots, N-2$. By Theorem 1.5 we have

$$E_{(-\infty,0)}(J_1) - E_{(-\infty,0)}(J_0) = \#_{[0,N-1]}(\psi_{0,+}, \psi_{1,-}) = \sum_{j=0}^{N-2} \#_j \tilde{\Phi}.$$

□

Theorem 5.8. *Let $a < 0$, then for all $n = 1, \dots, N-1$ we have*

$$\tilde{\Phi}_n - \tilde{\Phi}_{n-1} = (b_0(n) - b_1(n)) m_{0,+}(n+1) m_{1,-}(n-1). \tag{5.21}$$

Proof. By Lemma 5.3, (3.5), and Lemma 5.1 we have

$$\begin{aligned}\Phi_n - \Phi_{n-1} &= (W_n(\psi_{0,+}, \psi_{1,-}) - W_{n-1}(\psi_{0,+}, \psi_{1,-})) \prod_{i=1}^{N-2} -a(i) \\ &= (b_0(n) - b_1(n)) m_{0,+}(n+1) m_{1,-}(n-1).\end{aligned}$$

□

5.3 Proof of Sturm's theorem by Jacobi's theorem

In this section we present an alternative proof for (4.6), that is, we show that $E_{(-\infty,z)}(J) = \#_{(0,N)}(\psi_-(z))$ holds if $a(n) < 0$ for all n. Moreover, we extend this claim to the case $a(n) \neq 0$ for all n.
Therefore, let A be a Hermitian matrix of rank r and let

$$m_{A,-}(j) = \det(A_j) \tag{5.22}$$

be the leading principal minors of A, that is, A_j is the top left submatrix of A generated of the first j rows and columns of A. Moreover, we set $m_{A,-}(0) = 1$.

Theorem 5.9 (Jacobi)**.** *[8], Theorem 8.6.1 in [31]. If $m_{A,-}(j) \neq 0$ for all $j = 1, \ldots, r$, then,*

$$E_{(-\infty,0)}(A) = \#_{(0,r)}(m_{A,-}).$$

The proof is elementary. It was found in Jacobi's handwritten legacy and posthumously communicated by Borchardt [8] in 1857. In 1881 Gundelfinger [23] showed that the claim still holds if there are simple zeros in the sequence m_A:

Theorem 5.10 (Gundelfinger)**.** *[23], Theorem 8.6.2 in [31]. If the sequence $m_{A,-}(0), \ldots, m_{A,-}(r)$ contains no two successive zeros and $m_{A,-}(r) \neq 0$, then,*

$$E_{(-\infty,0)}(A) = \#_{(0,r)}(m_{A,-}).$$

The sequence $m_{A,-}$ changes sign around simple zeros.

Jacobi's theorem has moreover been extended to no three successive zeros in the sequence m_A which has been proven by Frobenius in [16].
For the next two lemmas we again relax our main assumption $a < 0$. It's enough to assume that $a(n) \neq 0$ for all n.

Lemma 5.11. *Let J be the Jacobi matrix from (1.8) where $a(n) \neq 0$ for all n. If $\det(J) = 0$, then $m_-(N-2) \neq 0$. If $m_-(j) = 0$ for some $j = 1, \ldots, N-2$, then*

$$m_-(j-1)m_-(j+1) < 0. \tag{5.23}$$

Proof. For all $j > 1$ by the Laplace expansion we have

$$m_-(j) = b(j)m_-(j-1) - a(j-1)^2 m_-(j-2), \tag{5.24}$$

hence the sequence m_- is a three-term-recurrence. Thus, if $m_-(j-1) = m_-(j) = 0$ for some j, then m_- vanishes which contradicts $m_-(0) = 1$. Moreover, if $m_-(j-1) = 0$, then $m_-(j-2)m_-(j) < 0$ holds by (5.24). □

Theorem 5.12. *Let J be the Jacobi matrix from (1.8) where $a(n) \neq 0$ for all n, then*

$$E_{(-\infty,0)}(J) = \#_{(0,N-1)}(m_-).$$

Proof. By Lemma 4.3 the spectrum of J is real and simple. If $0 \in \sigma(J)$, then $\det(J) = m_-(N-1) = 0$, hence $r = N-2$ and $m_-(r) \neq 0$ by Lemma 5.11. If $0 \notin \sigma(J)$ we have $\det(J) = m_-(N-1) = m_-(r) \neq 0$. In either case by Lemma 5.11 and Gundelfinger's theorem we have $E_{(-\infty,0)}(J) = \#_{(0,N-1)}(m_-)$. □

In [18], p. 79–85, Gantmacher and Krein considered classical oscillation theory for Jacobi matrices using the concept of Sturm chains and u-lines. In particular they established Theorem 5.12 in II.1.7°. Moreover, it can be found in [52, 5.38] and [21, Theorem 8.5.1] where it is deduced from the strict separation of the eigenvalues.

Theorem 5.13. *Let J be the Jacobi matrix from (1.8), $a(n) \neq 0$ for all n, and let $u_-(z)$ be a solution fulfilling $u_-(z,0) = 0$. Then,*

$$E_{(-\infty,z)}(J) = \#_{(0,N)}(u_-(z)) \tag{5.25}$$

$$= \#_{(0,N-1)}(m_{J-z,-}) \tag{5.26}$$

if we say $u_-(z)$ has a node at n if either

$$u_-(z,n) = 0 \quad \textit{or} \quad u_-(z,n)a(n)u_-(z,n+1) > 0. \tag{5.27}$$

The nodes of the minors $m_{J-z,-}$ are defined as usual, see (3.17).

Proof. The second claim follows from $\sigma(J) = \sigma(J-z) + z$ and Theorem 5.12 by $E_{(-\infty,z)}(J) = E_{(-\infty,0)}(J-z) = \#_{(0,N-1)}(m_{J-z,-})$.
Now look at the first claim: u_- is a constant multiple of ψ_- and hence they have equally many nodes. Moreover, by $\psi_-(z,0) = 0$ we have $\#_{(0,N)}(\psi_-) = \#_{(1,N)}(\psi_-)$. Compare the nodes of $m_{J-z,-}$ and $\psi_-(z)$: as in Lemma 5.1 we find

$$m_{J-z,-}(n-1) = \psi_-(z,n) \prod_{j=1}^{n-1} -a(j)^{-1} \tag{5.28}$$

The sequence $m_{J-z,-}$ has a node at n if either

$$m_{J-z,-}(n) = 0 \quad \text{or} \quad m_{J-z,-}(n)m_{J-z,-}(n+1) < 0 \tag{5.29}$$

holds and $m_{J-z,-}(n) = 0 \iff \psi_-(z,n+1) = 0$. Moreover, by

$$\begin{aligned} &m_{J-z,-}(n)m_{J-z,-}(n+1) \\ &\quad = -a(n+1)^{-1}\psi_-(z,n+1)\psi_-(z,n+2) \prod_{j=1}^{n} a(j)^{-2} \end{aligned} \tag{5.30}$$

we have $\#_{(0,N-1)}(m_{J-z,-}) = \#_{(1,N)}(\psi_-(z)) = \#_{(0,N)}(\psi_-(z))$ if we say $\psi_-(z)$ has a node at n if $\psi_-(z,n) = 0$ or $a(n)\psi_-(z,n)\psi_-(z,n+1) > 0$. □

For (5.27) confer also p. 3 of [46]. Of course, this theorem also extends to arbitrary tridiagonal matrices if we decompose the matrix according to (5.37) and consider each block separately.

5.4 A short note on Sylvester's criterion

It's well-known that the definiteness of a real symmetric matrix A can be read off the sign-pattern of the leading principal minors, confer e.g. [39]:

$$\begin{aligned} &A \text{ is positive definite} \\ &\quad \Longleftrightarrow \text{ all the upper left submatrices of } A \text{ have positive determinants,} \\ &A \text{ is positive semidefinite} \\ &\quad \Longrightarrow \text{ all the upper left submatrices of } A \text{ have nonnegative determinants.} \end{aligned} \tag{5.31}$$

That is, nonnegativity of the leading principal minors is a necessary but not sufficient criterion for $A \geqslant 0$. Therefore consider the striking counterexample given by the symmetric tridiagonal matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}, \quad x < 0, \tag{5.32}$$

which is frequently mentioned in the literature, confer e.g. [6, 7, 17, 18, 32]. Now look at the Jacobi matrix J from (1.8) where

$$a(n) \neq 0 \tag{5.33}$$

holds for all n and suppose that all the upper left submatrices $J(n)$ of J have nonnegative determinants

$$m_-(n) = \det(J(n)) \geqslant 0, \tag{5.34}$$

where we use the notation introduced in (5.3) and (5.5). We deduce from the Laplace expansion that

$$m_-(n) = b(n)m_-(n-1) - a(n-1)^2 m_-(n-2) \tag{5.35}$$

holds for all $n \geqslant 2$. Hence, no two consecutive minors vanish since (5.35) is a three-term recurrence (otherwise all of them would vanish, but we have $m_-(0) = 1$). Thus, if $m_-(n) = 0$ for any $1 \leqslant n < N-1$, then we obtain a contradiction from

$$0 < m_-(n+1) + a(n)^2 m_-(n-1) = b(n+1)m_-(n) = 0.$$

Hence, at most the determinant of J itself can vanish. If $\det(J) > 0$, then $J > 0$ by (5.31) and if $\det(J) = 0$, then $J(N-2) > 0$. Since J borders $J(N-2)$ the eigenvalues of J interlace those of $J(N-2)$, confer e.g. Theorem 4.3.8 in [24], hence by $0 \in \sigma(J)$ we have

Theorem 5.14. *Let J be a Jacobi matrix with $a(n) \neq 0$ for all n, then*

$$\begin{aligned} &J \text{ is positive semidefinite} \\ &\quad \iff \text{all the upper left submatrices of } J \text{ have nonnegative determinants} \end{aligned}$$

and if so, then at most the determinant of J vanishes.

We didn't find this claim in the literature, but it constitutes a special case of Theorem 5.12 which is Theorem 8.5.1 in [21].
Now, we drop the assumption $a(n) \neq 0$ and consider a tridiagonal matrix

$$T = \begin{pmatrix} b(1) & a(1) & & \\ a(1) & b(2) & \ddots & \\ & \ddots & \ddots & a(N-2) \\ & & a(N-2) & b(N-1) \end{pmatrix}. \tag{5.36}$$

Then, T is a direct sum of Jacobi matrices (i.e. matrices with non-zero secondary diagonals)

$$T = \oplus_k J_k \tag{5.37}$$

and the spectrum of T is the union of the spectra of the Jacobi matrices,

$$\sigma(T) = \cup_k \sigma(J_k). \tag{5.38}$$

Thus,

Theorem 5.15. *Let T be a tridiagonal matrix, then*

$$\begin{aligned} &T \text{ is positive semidefinite} \\ &\quad \iff \text{all the upper left submatrices of } J_k \text{ have nonnegative determinants} \\ &\qquad \text{for all } k, \end{aligned}$$

where J_k denote the Jacobi matrices such that $T = \oplus_k J_k$.

These findings can easily be extended to negative semidefinite tridiagonal matrices, therefore observe that

$$T \leqslant 0 \iff -T \geqslant 0$$

and, if we denote the leading principal minors of $-J$ by $m_{-J,-}(n)$, then

$$m_{-J,-}(n) = (-1)^n m_-(n).$$

Hence,

Theorem 5.16. *Let J be a Jacobi matrix with $a(n) \neq 0$ for all n, then*

$$\begin{aligned} &J \text{ is negative semidefinite} \\ &\quad\Longleftrightarrow \text{ the leading principal minors } m_-(0), \ldots, m_-(N-1) \text{ of } J \\ &\qquad\quad \text{alternate in sign.} \end{aligned}$$

If $J \leqslant 0$, then at most the determinant of J vanishes.

and

Theorem 5.17. *Let T be a tridiagonal matrix, then*

$$\begin{aligned} &T \text{ is negative semidefinite} \\ &\quad\Longleftrightarrow \text{ the leading principal minors } m_{J_k,-}(0), \ldots, m_{J_k,-}(N-1) \text{ of } J_k \\ &\qquad\quad \text{alternate in sign for all } k. \end{aligned}$$

If $T \geqslant 0$, then at most the determinants of the matrices J_k vanish.

Chapter 6

Triangle inequality and comparison theorem

We establish the triangle inequality and the comparison theorem for Wronskians which generalize Theorem 5.12 and Theorem 5.13 from [3] to different a's and will appear in [2]. Moreover, Theorem 6.3 generalizes and sharpens Theorem 5.11 from [3].

Theorem 6.1 (Comparison theorem for Wronskians I)**.** *Let $u_{-/+}$ denote a solution fulfilling the left/right Dirichlet boundary condition of J and let $J_1 \geqslant J_2$, then,*

$$\#_{[0,N]}(u_{0,\pm}(\lambda), u_{2,\mp}(\lambda)) \geqslant \#_{[0,N]}(u_{0,\pm}(\lambda), u_{1,\mp}(\lambda)), \tag{6.1}$$

where $\#_{[0,N]}$ can be replaced by $\#_{(0,N]}$, $\#_{[0,N)}$, or $\#_{(0,N)}$.

Proof. Let $\sigma(J_1) = \{\lambda_1, \dots, \lambda_{N-1}\}$ and $\sigma(J_2) = \{\tilde{\lambda}_1, \dots, \tilde{\lambda}_{N-1}\}$, then $\lambda_i \geqslant \tilde{\lambda}_i$ for all i by $J_1 \geqslant J_2$, cf. [31, Theorem 8.7.1], and hence we have $E_{(-\infty,\lambda)}(J_2) \geqslant E_{(-\infty,\lambda)}(J_1)$. Thus, by Theorem 1.5

$$\begin{aligned} \#_{[0,N]}(u_{0,+}(\lambda), u_{2,-}(\lambda)) &= E_{(-\infty,\lambda)}(J_2) - E_{(-\infty,\lambda)}(J_0) \\ &\geqslant E_{(-\infty,\lambda)}(J_1) - E_{(-\infty,\lambda)}(J_0) = \#_{[0,N]}(u_{0,+}(\lambda), u_{1,-}(\lambda)). \end{aligned} \tag{6.2}$$

The other claims follow analogously from $E_{(-\infty,\lambda]}(J_2) \geqslant E_{(-\infty,\lambda]}(J_1)$ and Theorem 1.5. □

Lemma 6.2. *Let $x, y \in \mathbb{R}$, then*

$$\lceil x \rceil + \lceil y \rceil - 1 \leq \lceil x + y \rceil \leq \lceil x \rceil + \lceil y \rceil, \tag{6.3}$$

$$\lceil x \rceil - \lceil y \rceil \leq \lceil x - y \rceil \leq \lceil x \rceil - \lceil y \rceil + 1, \tag{6.4}$$

$$\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1, \tag{6.5}$$

$$\lfloor x \rfloor - \lfloor y \rfloor - 1 \leq \lfloor x - y \rfloor \leq \lfloor x \rfloor - \lfloor y \rfloor. \tag{6.6}$$

Proof. Choose $k_x, k_y \in \mathbb{Z}, \chi, \psi \in (0, 1]$ such that $x = k_x + \chi$ and $y = k_y + \psi$, then $\lceil x \rceil = k_x + 1$ and $\lceil y \rceil = k_y + 1$ holds. Moreover,

$$\begin{aligned}\lceil x + y \rceil &= \lceil k_x + k_y + \chi + \psi \rceil \\ &= \begin{cases} k_x + k_y + 1 = \lceil x \rceil + \lceil y \rceil - 1 & \text{if } \chi + \psi \in (0, 1] \\ k_x + k_y + 2 = \lceil x \rceil + \lceil y \rceil & \text{if } \chi + \psi \in (1, 2] \end{cases}\end{aligned}$$

and

$$\begin{aligned}\lceil x - y \rceil &= \lceil k_x - k_y + \chi - \psi \rceil \\ &= \begin{cases} k_x - k_y = \lceil x \rceil - \lceil y \rceil & \text{if } \chi - \psi \in (-1, 0] \\ k_x - k_y + 1 = \lceil x \rceil - \lceil y \rceil + 1 & \text{if } \chi - \psi \in (0, 1). \end{cases}\end{aligned}$$

For the second claim choose $k_x, k_y \in \mathbb{Z}, \chi, \psi \in [0, 1)$ such that $x = k_x + \chi$ and $y = k_y + \psi$, then $\lfloor x \rfloor = k_x$ and $\lfloor y \rfloor = k_y$ holds. Moreover,

$$\begin{aligned}\lfloor x + y \rfloor &= \lfloor k_x + k_y + \chi + \psi \rfloor \\ &= \begin{cases} k_x + k_y = \lfloor x \rfloor + \lfloor y \rfloor & \text{if } \chi - \psi \in [0, 1) \\ k_x + k_y + 1 = \lfloor x \rfloor + \lfloor y \rfloor + 1 & \text{if } \chi - \psi \in [1, 2), \end{cases} \\ \lfloor x - y \rfloor &= \lfloor k_x - k_y + \chi - \psi \rfloor \\ &= \begin{cases} k_x - k_y - 1 = \lfloor x \rfloor - \lfloor y \rfloor - 1 & \text{if } \chi - \psi \in (-1, 0) \\ k_x - k_y = \lfloor x \rfloor - \lfloor y \rfloor & \text{if } \chi - \psi \in [0, 1). \end{cases}\end{aligned}$$

□

Theorem 6.3. *Let $m < n$, then*

$$|\#_{[m,n]}(u_0, u_1) - (\#_{(m,n)}(u_1) - \#_{(m,n)}(u_0))| \leq 1, \tag{6.7}$$

where $\#_{[m,n]}$ can be replaced by $\#_{(m,n]}$ or $\#_{[m,n)}$.

Proof. By Lemma 6.2 we have

$$0 \leq \lceil x - y \rceil - (\lceil x \rceil - \lceil y \rceil) \leq 1 \quad \text{and} \quad -1 \leq \lfloor x - y \rfloor - (\lfloor x \rfloor - \lfloor y \rfloor) \leq 0$$

for all $x, y \in \mathbb{R}$. Hence, by (3.46), Theorem 3.12, and $-\lceil x \rceil = \lfloor -x \rfloor$ we have

$$\begin{aligned}&|\#_{[m,n]}(u_0, u_1) - (\#_{(m,n)}(u_1) - \#_{(m,n)}(u_0))| \\ &\quad = |\lceil \Delta(n)/\pi \rceil - \lceil \Delta(m)/\pi \rceil \\ &\qquad - (\lceil \theta_1(n)/\pi \rceil - \lfloor \theta_1(m)/\pi \rfloor - \lceil \theta_0(n)/\pi \rceil + \lfloor \theta_0(m)/\pi \rfloor)| \\ &\quad = |\lceil (\theta_1(n) - \theta_0(n))/\pi \rceil - (\lceil \theta_1(n)/\pi \rceil - \lceil \theta_0(n)/\pi \rceil)\end{aligned}$$

$$+ \lfloor (\theta_0(m) - \theta_1(m))/\pi \rfloor - (\lfloor \theta_0(m)/\pi \rfloor - \lfloor \theta_1(m)/\pi \rfloor)| \leq 1.$$

By Lemma 3.19 and Theorem 3.12 we moreover have

$$\begin{aligned}
&\#_{(m,n]}(u_0, u_1) - (\#_{(m,n)}(u_1) - \#_{(m,n)}(u_0)) \\
&\quad = \lceil \Delta(n)/\pi \rceil - \lfloor \Delta(m)/\pi \rfloor - 1 - \lceil \theta_1(n)/\pi \rceil + \lfloor \theta_1(m)/\pi \rfloor + \lceil \theta_0(n)/\pi \rceil \\
&\quad\quad - \lfloor \theta_0(m)/\pi \rfloor \\
&\quad = \lceil \Delta(n)/\pi \rceil - (\lceil \theta_1(n)/\pi \rceil - \lceil \theta_0(n)/\pi \rceil) \\
&\quad\quad - (\lfloor \Delta(m)/\pi \rfloor - (\lfloor \theta_1(m)/\pi \rfloor - \lfloor \theta_0(m)/\pi \rfloor)) - 1 \\
&\#_{[m,n)}(u_0, u_1) - (\#_{(m,n)}(u_1) - \#_{(m,n)}(u_0)) \\
&\quad = \lfloor \Delta(n)/\pi \rfloor - \lceil \Delta(m)/\pi \rceil + 1 - \lceil \theta_1(n)/\pi \rceil + \lfloor \theta_1(m)/\pi \rfloor + \lceil \theta_0(n)/\pi \rceil \\
&\quad\quad - \lfloor \theta_0(m)/\pi \rfloor \\
&\quad = 1 + \lfloor (\theta_0(m) - \theta_1(m))/\pi \rfloor - (\lfloor \theta_0(m)/\pi \rfloor - \lfloor \theta_1(m)/\pi \rfloor) \\
&\quad\quad - (\lceil (\theta_0(n) - \theta_1(n))/\pi \rceil - (\lceil \theta_0(n)/\pi \rceil - \lceil \theta_1(n)/\pi \rceil)).
\end{aligned}$$

□

Theorem 6.4 (Triangle inequality for Wronskians)**.** *Confer [3]. We have*

$$|\#_{[m,n]}(u_0, u_2) - (\#_{[m,n]}(u_0, u_1) + \#_{[m,n]}(u_1, u_2))| \leq 1, \tag{6.8}$$

where $\#_{[m,n]}$ *can be replaced by* $\#_{(m,n]}$ *and* u_j *be solutions of* $\tau_j u_j = \lambda u_j, j = 0, 1, 2$.

Proof. Abbreviate $\Delta_{i,j} = \Delta_{u_i,u_j}$, then $\Delta_{0,1} + \Delta_{1,2} = \Delta_{0,2}$. By (3.46) we have

$$\#_{[m,n]}(u_0, u_2) = \lceil \Delta_{0,2}(n)/\pi \rceil - \lceil \Delta_{0,2}(m)/\pi \rceil,$$

hence

$$\begin{aligned}
&\#_{[m,n]}(u_0, u_1) + \#_{[m,n]}(u_1, u_2) \\
&\quad = \lceil \Delta_{0,1}(n)/\pi \rceil + \lceil \Delta_{1,2}(n)/\pi \rceil - (\lceil \Delta_{0,1}(m)/\pi \rceil + \lceil \Delta_{1,2}(m)/\pi \rceil) \\
&\quad \leq \lceil \Delta_{0,2}(n)/\pi \rceil + 1 - \lceil \Delta_{0,2}(m)/\pi \rceil = \#_{[m,n]}(u_0, u_2) + 1
\end{aligned}$$

and

$$\begin{aligned}
&\#_{[m,n]}(u_0, u_1) + \#_{[m,n]}(u_1, u_2) \\
&\quad \geq \lceil \Delta_{0,2}(n)/\pi \rceil - (\lceil \Delta_{0,2}(m)/\pi \rceil + 1) = \#_{[m,n]}(u_0, u_2) - 1
\end{aligned}$$

holds by $\lceil x + y \rceil \leq \lceil x \rceil + \lceil y \rceil \leq \lceil x + y \rceil + 1$ for all $x, y \in \mathbb{R}$. Further, by

Lemma 3.19 and $\lfloor x+y \rfloor - 1 \leqslant \lfloor x \rfloor + \lfloor y \rfloor \leqslant \lfloor x+y \rfloor$ we now have

$$\begin{aligned}
&\#_{(m,n]}(u_0,u_1) + \#_{(m,n]}(u_1,u_2) \\
&\quad = \lceil \Delta_{0,1}(n)/\pi \rceil - \lfloor \Delta_{0,1}(m)/\pi \rfloor - 1 + \lceil \Delta_{1,2}(n)/\pi \rceil - \lfloor \Delta_{1,2}(m)/\pi \rfloor - 1 \\
&\quad \leq \lceil \Delta_{0,2}(n)/\pi \rceil - \lfloor \Delta_{0,2}(m)/\pi \rfloor = \#_{(m,n]}(u_0,u_2) + 1
\end{aligned}$$

and $\#_{(m,n]}(u_0,u_2) \leq \#_{(m,n]}(u_0,u_1) + \#_{(m,n]}(u_1,u_2) + 1$. □

Theorem 6.5 (Comparison theorem for Wronskians II)**.** *If either*

A $W_j(u_0,u_1)u_0(j+1)u_1(j+1) \leqslant 0$ *and* $W_j(u_1,u_2)u_1(j+1)u_2(j+1) \leqslant 0$ *for all* $j = 0,\dots,N-2$ *or*

B $a_0 = a_1 = a_2$ *and* $b_0(j) \geqslant b_1(j) \geqslant b_2(j)$ *for all* $j = 1,\dots N-1$

holds and 0 *and* $N-2$ *are (positive) nodes of* $W(u_0,u_1)$*, then* $W(u_0,u_2)$ *has at least one positive node at* $0,\dots,N-2$*.*

Proof. In either case we have $\#_j(u_0,u_1) \geqslant 0$ and $\#_j(u_1,u_2) \geqslant 0$ for all $j = 0,\dots,N-2$ and hence from Theorem 6.4 we conclude

$$\#_{[0,N-1]}(u_0,u_2) \geqslant \underbrace{\#_{[0,N-1]}(u_0,u_1)}_{\geqslant 2} + \underbrace{\#_{[0,N-1]}(u_1,u_2)}_{\geqslant 0} - 1.$$

□

Chapter 7

Criteria for the oscillation of the Wronskian

In this chapter we show that the number of nodes of the Wronskian on the half-line and on the line is finite in a gap of the essential spectrum. From now on let $u_j(\lambda_j)$ be solutions of $(\tau_j - \lambda_j)u_j = 0$, where $j = 0, 1, 2$, and

$$a = a_0 = a_1 = a_2. \tag{7.1}$$

Definition 7.1. *We call a perturbation $\Delta b = b_0 - b_1$* sign-definite *at z (near ∞) if there exists an N such that either $\Delta b(n) \geqslant z$ or $\Delta b(n) \leqslant z$ holds for all $n > N$. Moreover, we say Δb is* sign-definite *if Δb is sign-definite for all $z \in \mathbb{R}$.*

If $b_0 - b_1$ is sign-definite at $\lambda_0 - \lambda_1$, then $b_1 - b_0$ is sign-definite at $\lambda_1 - \lambda_0$ and

$$\limsup_{n\to\infty} \sum_{j=0}^{n} \#_j(u_0, u_1) = \liminf_{n\to\infty} \sum_{j=0}^{n} \#_j(u_0, u_1), \tag{7.2}$$

$$\limsup_{n\to\infty} \sum_{j=0}^{n} \#_j(u_1, u_0) = \liminf_{n\to\infty} \sum_{j=0}^{n} \#_j(u_1, u_0) \tag{7.3}$$

holds. If so, then either

$$\#_{[0,\infty]}(u_0, u_1) = \sum_{j=0}^{\infty} \#_j(u_0, u_1) = k \in \mathbb{Z} \tag{7.4}$$

holds and there exists an N such that $\#_n(u_0, u_1) = 0$ for all $n > N$ or

$$\#_{[0,\infty]}(u_0, u_1) = \sum_{j=0}^{\infty} \#_j(u_0, u_1) = \pm\infty \tag{7.5}$$

holds and for all N there exists an $n > N$ such that $\#_n(u_0, u_1) = \pm 1$. By the triangle inequality (Theorem 6.4) we have $|\#_{[0,n]}(u_0, u_1) + \#_{[0,n]}(u_1, u_0)| \leqslant 1$ for all n, hence

$$\#_{[0,\infty]}(u_0, u_1) \text{ is finite} \quad \Longleftrightarrow \quad \#_{[0,\infty]}(u_1, u_0) \quad \text{is finite,} \tag{7.6}$$

$$\#_{[0,\infty]}(u_0, u_1) \text{ is finite} \quad \Longleftrightarrow \quad \#_{[0,\infty]}(\tilde{u}_0, u_1) \quad \text{is finite} \tag{7.7}$$

for all solutions $\tilde{u}_0$ of $(\tau_0 - \lambda_0)\tilde{u}_0 = 0$. So the following is well-defined:

Definition 7.2. *Let $b_0 - b_1$ be sign-definite at $\lambda_0 - \lambda_1$ near ∞, then we call $\tau_0 - \lambda_0$ and $\tau_1 - \lambda_1$* relatively nonoscillatory near ∞ *and denote*

$$\tau_0 - \lambda_0 \overset{rno_+}{\sim} \tau_1 - \lambda_1 \quad \text{if} \quad \sum_{j=0}^{\infty} \#_j(u_0, u_1) \quad \text{is finite}$$

for one (and hence for all) solutions of $(\tau_j - \lambda_j)u_j = 0$, $j = 0, 1$. Otherwise we call $\tau_0 - \lambda_0$ and $\tau_1 - \lambda_1$ relatively oscillatory near ∞.

Here, we only carried out the $+\infty$-case. Obviously we obtain the same results near $-\infty$ if $b_0 - b_1$ is sign-definite at $\lambda_0 - \lambda_1$ near $-\infty$. If so, then we define analogously

$$\tau_0 - \lambda_0 \overset{rno_-}{\sim} \tau_1 - \lambda_1 \quad \text{if } \#_{[-\infty,0]}(u_0, u_1) = \sum_{j=-\infty}^{-1} \#_j(u_0, u_1) \quad \text{is finite,} \tag{7.8}$$

$$\tau_0 - \lambda_0 \overset{rno}{\sim} \tau_1 - \lambda_1 \quad \text{if } \#_{[-\infty,\infty]}(u_0, u_1) = \sum_{j=-\infty}^{\infty} \#_j(u_0, u_1) \quad \text{is finite.} \tag{7.9}$$

Lemma 7.3. *Let $b_0 - b_1, b_1 - b_2$ and $b_0 - b_2$ be sign-definite at 0 near $\pm\infty$, then*

$$\tau_0 \overset{rno_\pm}{\sim} \tau_1, \tau_1 \overset{rno_\pm}{\sim} \tau_2 \implies \tau_0 \overset{rno_\pm}{\sim} \tau_2 \tag{7.10}$$

If moreover $b_0 \geqslant b_1 \geqslant b_2$ near $\pm\infty$, then

$$\tau_0 \overset{rno_\pm}{\sim} \tau_2 \implies \tau_0 \overset{rno_\pm}{\sim} \tau_1, \tau_1 \overset{rno_\pm}{\sim} \tau_2. \tag{7.11}$$

Proof. We have $|\#_{[0,n]}(u_0, u_2) - (\#_{[0,n]}(u_0, u_1) + \#_{[0,n]}(u_1, u_2))| \leq 1$ for all n by the triangle inequality. If $b_0 \geqslant b_1 \geqslant b_2$, then the nodes of the Wronskian are weighted positive near ∞. □

Lemma 7.4. *If $\tau_0 - \lambda_0 \overset{rno_+}{\sim} \tau_1 - \lambda_1$, then there exists an N such that*

$$W_n(u_0, u_1) > 0, \quad W_n(u_0, u_1) < 0, \text{ or } \quad W_n(u_0, u_1) = 0$$

holds for all $n > N$, i.e. the Wronskian is of one sign near ∞.
The same holds near $-\infty$ if $\tau_0 - \lambda_0 \overset{rno_-}{\sim} \tau_1 - \lambda_1$.

Proof. If $W_n(u_0(\lambda_0), u_1(\lambda_1)) = 0$ for all $n > \tilde{N}$ the claim holds obviously. If not, then by $\tau_0 - \lambda_0 \overset{rno_+}{\sim} \tau_1 - \lambda_1$ there exists an $N \in \mathbb{N}$ such that $W_N(u_0, u_1) \neq 0$, $\#_n(u_0, u_1) = 0$ and $b_0(n) - \lambda_0 - b_1(n) + \lambda_1$ is of one sign for all $n \geqslant N$. The Wronskian cannot change sign at some $m \geqslant N$. Moreover, $W(u_0, u_1)$ cannot vanish at some interval $m, \dots, n$, where $m > N, n \geqslant m$, since if so, the Wronskian has a node either at the beginning or at the end of the interval since $b_0 - \lambda_0 - b_1 + \lambda_1$ is of one sign. Analogously, for the $-\infty$-case. □

Thus, if $\tau_0 - \lambda_0 \overset{rno}{\sim} \tau_1 - \lambda_1$, then the following limits exist:

$$\begin{aligned}
\#_{(-\infty,\infty]}(u_0, u_1) &= \lim_{n\to\infty} \#_{(-n,n]}(u_0, u_1) \qquad (7.12)\\
&= \#_{[-\infty,\infty]}(u_0, u_1) - \begin{cases} 1 & \text{if } W(u_0, u_1) \equiv 0 \text{ near } -\infty \\ 0 & \text{otherwise,} \end{cases}\\
\#_{[-\infty,\infty)}(u_0, u_1) &= \lim_{n\to\infty} \#_{[-n,n)}(u_0, u_1) \qquad (7.13)\\
&= \#_{[-\infty,\infty]}(u_0, u_1) + \begin{cases} 1 & \text{if } W(u_0, u_1) \equiv 0 \text{ near } \infty \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\#_{(-\infty,\infty)}(u_0, u_1) &= \lim_{n\to\infty} \#_{(-n,n)}(u_0, u_1) \qquad (7.14)\\
&= \#_{[-\infty,\infty]}(u_0, u_1) - \begin{cases} 1 & \text{if } W_0(u_0, u_1) \equiv 0 \text{ near } -\infty \\ 0 & \text{otherwise} \end{cases}\\
&\quad + \begin{cases} 1 & \text{if } W(u_0, u_1) \equiv 0 \text{ near } \infty \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

and analogously for a finite endpoint

$$\#_{(0,\infty]}(u_0, u_1) = \lim_{n\to\infty} \#_{(0,n]}(u_0, u_1), \qquad (7.15)$$

$$\#_{[0,\infty)}(u_0, u_1) = \lim_{n\to\infty} \#_{[0,n)}(u_0, u_1), \qquad (7.16)$$

$$\#_{(0,\infty)}(u_0, u_1) = \lim_{n\to\infty} \#_{(0,n)}(u_0, u_1). \qquad (7.17)$$

If $\lim_{n\to\infty} \Delta b(n) = z$, then Δb is sign-definite if it is sign-definite at z. We abbreviate

$$b_0 \downarrow b_1 \qquad (7.18)$$

(or $b_0 \uparrow b_1$) near $\pm\infty$ whenever $\lim_{n\to\infty} \Delta b(n) = 0$ and $\Delta b \geqslant 0$ (or $\Delta b \leqslant 0$) holds near $\pm\infty$.

Remark 7.5. *If $W(u_0, u_1)$ vanishes at some interval $m, \dots, n$, then it could* *be possible that* $\#_{m-1}(u_0, u_1) = 0$, $\#_n(u_0, u_1) = 0$ *and* $\#_{m-1}(u_1, u_0) = -1$, $\#_n(u_1, u_0) = 1$. *Hence, if* Δb *oscillates and* $\#_{[0,\infty]}(u_0, u_1)$ *exists, then the sum* $\#_{[0,\infty]}(u_1, u_0)$ *doesn't have to exist, i.e. we could have*

$$\limsup_{n\to\infty} \#_{[0,n]}(u_1, u_0) \neq \liminf_{n\to\infty} \#_{[0,n]}(u_1, u_0). \tag{7.19}$$

This could also happen near $-\infty$. *Thus, to obtain our main theorems we assume that the perturbation is sign-definite near* $+\infty$ *and near* $-\infty$, *hence only the case where* $b_0 \downarrow b_1$ *or* $b_0 \uparrow b_1$ *holds is considered in the sequel, although some claims also hold if we just assume* $b_0 \to b_1$ *provided the limits exist.*

Lemma 7.6. *Let* $b_0 \downarrow b_1$ *or* $b_0 \uparrow b_1$ *near* $\pm\infty$ *and* $\tau_0 - \lambda_0 \overset{rno_\pm}{\sim} \tau_1 - \lambda_1$.
If $\lambda_0 \neq \lambda_1$, *then there exists an* N *such that*

$$W_n(u_0(\lambda_0), u_1(\lambda_1)) \neq 0 \tag{7.20}$$

holds for all $\pm n > N$.
If $\lambda_0 = \lambda_1$, *then there exists an* N *such that either*

- u_0 *and* u_1 *are linearly independent near* $\pm\infty$ *and* $W_n(u_0, u_1) \neq 0$ *or*
- u_0 *and* u_1 *are linearly dependent near* $\pm\infty$ *and* $W_n(u_0, u_1) = 0$

for all $\pm n > N$.

Proof. Let $\lambda_0 \neq \lambda_1$ and suppose the claim does not hold, then by Lemma 7.4 there exists an N such that $W_m(u_0, u_1) = 0$ for all $m \geqslant N$. Then, by (3.14) we have either

$$b_0(m) - b_1(m) = \lambda_0 - \lambda_1 \quad \text{or} \quad u_0(m) = u_1(m) = 0 \tag{7.21}$$

for all $m > N$ which contradicts $\lim_{n\to\infty}(b_0 - b_1)(n) = 0$ since the zeros of u_0 are simple and $\lambda_0 - \lambda_1 \neq 0$. □

Lemma 7.7. *Let* $b_0 \downarrow b_1$ *or* $b_0 \uparrow b_1$ *near* $\pm\infty$ *and let* $\lambda_0 \neq \lambda_1$. *If* $\tau_0 - \lambda_0 \overset{rno_\pm}{\sim} \tau_1 - \lambda_1$, *then the solutions* $u_0(\lambda_0)$ *and* $u_1(\lambda_1)$ *are linearly independent near* $\pm\infty$ *and have at most finitely many common zeros near* $\pm\infty$.

Proof. Suppose there are infinitely many points $j \in \mathbb{N}$ such that $u_0(\lambda_0, j) = u_1(\lambda_1, j) = 0$, then $W_j(u_0, u_1) = a(j)(u_0(j)u_1(j+1) - u_1(j)u_0(j+1)) = 0$, which contradicts Lemma 7.6. The same holds near $-\infty$. □

Recall the following

Theorem 7.8. *[42, Cor 4.18, Cor. 4.20]. Let* $\lambda_0 < \lambda_1$, *then*

$$\mathrm{tr}(P_{(\lambda_0,\lambda_1)}(H_\pm)) < \infty \quad \Longleftrightarrow \quad \tau - \lambda_0 \overset{rno_\pm}{\sim} \tau - \lambda_1, \tag{7.22}$$

$$\operatorname{tr}(P_{(\lambda_0,\lambda_1)}(H)) < \infty \quad \Longleftrightarrow \quad \tau - \lambda_0 \overset{rno}{\sim} \tau - \lambda_1. \tag{7.23}$$

As a small application thereof we notice the following

Theorem 7.9. *Let $\operatorname{tr}(P_{(z_-,z_+)}(H_+)) < \infty$, $z_- < z_+$, and let H_n be the leading principle submatrices of the semi-infinite Jacobi operator H_+. Then, there are at most finitely many n such that z_- and z_+ both are eigenvalues of H_n.*

Proof. The solutions $u_-(z_-)$ and $u_-(z_+)$ have at most finitely many common zeros near ∞ by Lemma 7.7. □

Finally, we now obtain the main findings of this chapter, namely criteria for the finiteness of the number of nodes of the Wronskian. Therefore we consecutively investigate the possible Wronskians on the half-line and on the line.

Theorem 7.10. *If $b_0 \downarrow b_1$ or $b_0 \uparrow b_1$ near $\pm\infty, \lambda, \lambda_0, \lambda_1 \notin \sigma_{ess}(H^0_\pm)$, and $\lambda_0 < \lambda_1$, then*

$$\tau_0 - \lambda \overset{rno_\pm}{\sim} \tau_1 - \lambda \tag{7.24}$$

and

$$\tau_0 - \lambda_0 \overset{rno_\pm}{\sim} \tau_1 - \lambda_1 \quad \Longleftrightarrow \quad [\lambda_0, \lambda_1] \cap \sigma_{ess}(H^0_\pm) = \emptyset. \tag{7.25}$$

Proof. Let $b_0 \downarrow b_1$ near ∞, then, since the essential spectrum is a closed subset of $\mathbb{R}$, we have $[\lambda, \lambda+\varepsilon] \cap \sigma_{ess}(H^0_+) = \emptyset$ for some $\varepsilon > 0$. Hence, by Theorem 7.8 we have $\tau_0 - \lambda \overset{rno_+}{\sim} \tau_0 - (\lambda+\varepsilon)$. Moreover, $b_0 - \lambda \geqslant b_1 - \lambda \geqslant b_0 - (\lambda+\varepsilon)$ holds near ∞ and hence by (7.11) we have $\tau_0 - \lambda \overset{rno_+}{\sim} \tau_1 - \lambda$. For the $b_0 \uparrow b_1$-case just interchange τ_0 and τ_1. Clearly, the same holds near $-\infty$.
Now, consider the second claim: by Theorem 7.8 we have $[\lambda_0, \lambda_1] \cap \sigma_{ess}(H^0_+) = \emptyset \implies \tau_0 - \lambda_0 \overset{rno_+}{\sim} \tau_0 - \lambda_1$. Moreover, by (7.24) we have $\lambda_1 \notin \sigma_{ess}(H^0_+) \implies \tau_0 - \lambda_1 \overset{rno_+}{\sim} \tau_1 - \lambda_1$. Hence, by (7.10) we have $\tau_0 - \lambda_0 \overset{rno_+}{\sim} \tau_1 - \lambda_1$. On the other hand, suppose $\tau_0 - \lambda_0 \overset{rno_+}{\sim} \tau_1 - \lambda_1$ holds, then by $\lambda_1 \notin \sigma_{ess}(H^0_+)$ and by (7.24) we have $\tau_1 - \lambda_1 \overset{rno_+}{\sim} \tau_0 - \lambda_1$. Hence, again by (7.10) we have $\tau_0 - \lambda_0 \overset{rno_+}{\sim} \tau_0 - \lambda_1$. Thus, Theorem 7.8 implies $\operatorname{tr}(P_{(\lambda_0,\lambda_1)}(H^0_+)) < \infty$ and $\lambda_0, \lambda_1 \notin \sigma_{ess}(H^0_+)$ proves the claim. The same holds near $-\infty$. □

Theorem 7.11. *If $b_0 \downarrow b_1$ or $b_0 \uparrow b_1$ holds near ∞ and $b_0 \downarrow b_1$ or $b_0 \uparrow b_1$ holds near $-\infty$, $\lambda_0 < \lambda_1$, and $\lambda, \lambda_0, \lambda_1 \notin \sigma_{ess}(H_0)$, then*

$$\tau_0 - \lambda \overset{rno}{\sim} \tau_1 - \lambda \tag{7.26}$$

and

$$\tau_0 - \lambda_0 \overset{rno}{\sim} \tau_1 - \lambda_1 \quad \Longleftrightarrow \quad [\lambda_0, \lambda_1] \cap \sigma_{ess}(H_0) = \emptyset. \tag{7.27}$$

Proof. By $\sigma_{ess}(H_0) = \sigma_{ess}(H^0_-) \cup \sigma_{ess}(H^0_+)$ and (7.24) we have $\tau_0 - \lambda \overset{rno_\pm}{\sim} \tau_1 - \lambda$, hence the first claim holds. If $\tau_0 - \lambda_0 \overset{rno}{\sim} \tau_1 - \lambda_1$ holds, then $\tau_0 - \lambda_0 \overset{rno_+}{\sim} \tau_1 - \lambda_1$ and $\tau_0 - \lambda_0 \overset{rno_-}{\sim} \tau_1 - \lambda_1$ hold. Thus, $[\lambda_0, \lambda_1] \cap \sigma_{ess}(H^0_\pm) = \emptyset$ holds by Theorem 7.10 and hence again by $\sigma_{ess}(H_0) = \sigma_{ess}(H_-^0) \cup \sigma_{ess}(H_+^0)$ we have $[\lambda_0, \lambda_1] \cap \sigma_{ess}(H_0) = \emptyset$. On the other hand, if $[\lambda_0, \lambda_1] \cap \sigma_{ess}(H_0) = \emptyset$ holds, then clearly again by Theorem 7.10 the second claim holds. $\square$

We remark that $\lambda \in \sigma_{ess}(H_0)$ does not imply that $\tau_0 - \lambda$ and $\tau_1 - \lambda$ are relatively oscillatory since we actually have $\tau_0 - \lambda \overset{rno}{\sim} \tau_0 - \lambda$. In the next step we consider spectral intervals with boundaries attaining the essential spectrum.

Theorem 7.12. *Let $b_0 \downarrow b_1$ or $b_0 \uparrow b_1$ near $\pm\infty$ and let $\underline{\lambda} < \overline{\lambda}$.*
If $\operatorname{tr} P_{(\underline{\lambda},\overline{\lambda})}(H_\pm^0) + \operatorname{tr} P_{(\underline{\lambda},\overline{\lambda})}(H_\pm^1) < \infty$, then

$$\tau_0 - \underline{\lambda} \overset{rno_\pm}{\sim} \tau_1 - \overline{\lambda} \quad \textit{and} \quad \tau_0 - \overline{\lambda} \overset{rno_\pm}{\sim} \tau_1 - \underline{\lambda}. \tag{7.28}$$

If $b_0 \downarrow b_1$ near $\pm\infty$, then

$$\tau_0 - \underline{\lambda} \overset{rno_\pm}{\sim} \tau_1 - \overline{\lambda} \quad \Longleftrightarrow \quad \operatorname{tr} P_{(\underline{\lambda},\overline{\lambda})}(H_\pm^0) + \operatorname{tr} P_{(\underline{\lambda},\overline{\lambda})}(H_\pm^1) < \infty. \tag{7.29}$$

If $b_0 \uparrow b_1$ near $\pm\infty$, then

$$\tau_0 - \overline{\lambda} \overset{rno_\pm}{\sim} \tau_1 - \underline{\lambda} \quad \Longleftrightarrow \quad \operatorname{tr} P_{(\underline{\lambda},\overline{\lambda})}(H_\pm^0) + \operatorname{tr} P_{(\underline{\lambda},\overline{\lambda})}(H_\pm^1) < \infty. \tag{7.30}$$

Proof. Let $b_0 \downarrow b_1$, then $b_0 \geqslant b_1 \geqslant b_0 - (\overline{\lambda} - \underline{\lambda}) \geqslant b_1 - (\overline{\lambda} - \underline{\lambda})$ near $\pm\infty$, hence

$$b_0 - \underline{\lambda} \geqslant b_1 - \underline{\lambda} \geqslant b_0 - \overline{\lambda} \geqslant b_1 - \overline{\lambda} \tag{7.31}$$

near $\pm\infty$. Suppose we have $\operatorname{tr} P_{(\underline{\lambda},\overline{\lambda})}(H_\pm^0) + \operatorname{tr} P_{(\underline{\lambda},\overline{\lambda})}(H_\pm^1) < \infty$, then $\tau_0 - \underline{\lambda} \overset{rno_\pm}{\sim} \tau_0 - \overline{\lambda}$ and $\tau_1 - \underline{\lambda} \overset{rno_\pm}{\sim} \tau_1 - \overline{\lambda}$ holds by Theorem 7.8. Hence, by (7.11) we have

$$\tau_0 - \underline{\lambda} \overset{rno_\pm}{\sim} \tau_1 - \underline{\lambda} \overset{rno_\pm}{\sim} \tau_0 - \overline{\lambda} \overset{rno_\pm}{\sim} \tau_1 - \overline{\lambda} \tag{7.32}$$

and thus (7.10) proves the claim. On the other hand, if $\tau_0 - \underline{\lambda} \overset{rno_\pm}{\sim} \tau_1 - \overline{\lambda}$ holds, then by (7.31) and (7.11) we have $\tau_0 - \underline{\lambda} \overset{rno_\pm}{\sim} \tau_0 - \overline{\lambda}$ and $\tau_1 - \underline{\lambda} \overset{rno_\pm}{\sim} \tau_1 - \overline{\lambda}$, thus the claim follows from Theorem 7.8. For the $b_0 \uparrow b_1$-case just interchange τ_0 and τ_1. $\square$

Now, we're ready for a proof of Theorem 1.3:

Theorem 7.13. *Let $b_0 \downarrow b_1$ or $b_0 \uparrow b_1$ and let $\underline{\lambda} < \overline{\lambda}$.*
If $\operatorname{tr} P_{(\underline{\lambda},\overline{\lambda})}(H_0) + \operatorname{tr} P_{(\underline{\lambda},\overline{\lambda})}(H_1) < \infty$, then

$$\tau_0 - \underline{\lambda} \overset{rno}{\sim} \tau_1 - \overline{\lambda} \quad \textit{and} \quad \tau_0 - \overline{\lambda} \overset{rno}{\sim} \tau_1 - \underline{\lambda}. \tag{7.33}$$

If $b_0 \downarrow b_1$, then

$$\tau_0 - \underline{\lambda} \overset{rno}{\sim} \tau_1 - \overline{\lambda} \quad \Longleftrightarrow \quad \operatorname{tr} P_{(\underline{\lambda},\overline{\lambda})}(H_0) + \operatorname{tr} P_{(\underline{\lambda},\overline{\lambda})}(H_1) < \infty. \tag{7.34}$$

If $b_0 \uparrow b_1$*, then*

$$\tau_0 - \overline{\lambda} \overset{rno}{\sim} \tau_1 - \underline{\lambda} \quad \iff \quad \operatorname{tr} P_{(\underline{\lambda},\overline{\lambda})}(H_0) + \operatorname{tr} P_{(\underline{\lambda},\overline{\lambda})}(H_1) < \infty. \tag{7.35}$$

Proof. Let $b_0 \downarrow b_1$ or $b_0 \uparrow b_1$ and $\operatorname{tr} P_{(\underline{\lambda},\overline{\lambda})}(H_0) + \operatorname{tr} P_{(\underline{\lambda},\overline{\lambda})}(H_1) < \infty$, then by Theorem 7.8 we have $\operatorname{tr}(P_{(\lambda_0,\lambda_1)}(H_0)) < \infty \implies \tau_0 - \lambda_0 \overset{rno}{\sim} \tau_0 - \lambda_1$, thus $\tau_0 - \lambda_0 \overset{rno_\pm}{\sim} \tau - \lambda_1$. Hence, $\operatorname{tr}(P_{(\lambda_0,\lambda_1)}(H^0_\pm)) < \infty$. The same holds for H_1. Thus, the first claim holds by Theorem 7.12.
If $b_0 \downarrow b_1$, then $\tau_0 - \underline{\lambda} \overset{rno}{\sim} \tau_1 - \overline{\lambda}$ implies $\tau_0 - \underline{\lambda} \overset{rno_\pm}{\sim} \tau_1 - \overline{\lambda}$ and hence by Theorem 7.12 we have $\operatorname{tr} P_{(\underline{\lambda},\overline{\lambda})}(H^0_\pm) + \operatorname{tr} P_{(\underline{\lambda},\overline{\lambda})}(H^1_\pm) < \infty$. Now, we conclude from Theorem 7.8 that $\operatorname{tr} P_{(\underline{\lambda},\overline{\lambda})}(H_0) + \operatorname{tr} P_{(\underline{\lambda},\overline{\lambda})}(H_1) < \infty$. This proves the second claim. To obtain the third claim just interchange τ_0 and τ_1. □

Remark 7.14. *Let* $\lambda_0, \lambda_1 \in \Omega = \mathbb{R} \setminus \sigma_{ess}(H^0_\pm)$ *and* $b_0 \downarrow b_1$ *or* $b_0 \uparrow b_1$ *near* $\pm\infty$*. Then,* $\tau_0 - \lambda_0 \overset{rno_\pm}{\sim} \tau_1 - \lambda_1$ *iff* λ_0 *and* λ_1 *are in the same connected component of* Ω*.*

Chapter 8

Approximation

In this chapter we approximate Jacobi operators on the half-line (and their Weyl solutions u_+) by finite Jacobi matrices (and solutions fulfilling a Dirichlet boundary condition on the right-hand side). To simplify notation we use semi-infinite matrices instead of finite matrices to approximate Jacobi operators on $\mathbb{Z}$.

8.1 ... of infinite matrices and their spectra

At first we show how to alter a boundary condition of a finite (or a semi-infinite) Jacobi operator such that it is then fulfilled by a particular Weyl solution. This doesn't have to be possible for all indices n, hence we choose a suitable index set $\mathscr{I}_v$. Therefore, let $v \in \ell(\mathbb{N})$ such that

$$\mathscr{I}_v = \{n \in \mathbb{N}, n > 2 \mid v(n-1) \neq 0\} \tag{8.1}$$

is an infinite set and let $b_v \in \ell(\mathbb{N})$,

$$b_v(n-1) = \begin{cases} \frac{a(n-1)v(n)}{v(n-1)} & \text{if } n \in \mathscr{I}_v \\ 0 & \text{otherwise.} \end{cases} \tag{8.2}$$

Analogously, let $w \in \ell(\mathbb{N})$ such that

$$\mathscr{I}_w = \{m \in -\mathbb{N}, m < -2 \mid w(m+1) \neq 0\} \tag{8.3}$$

is an infinite set and let $b_w \in \ell(-\mathbb{N})$,

$$b_w(m+1) = \begin{cases} \frac{a(m)w(m)}{w(m+1)} & \text{if } m \in \mathscr{I}_w \\ 0 & \text{otherwise.} \end{cases} \tag{8.4}$$

Now, we alter the Jacobi matrices introduced in (2.43) and (2.44) such that

$$H_n^v = H_{0,n} + \operatorname{diag}(b_v(n-1)\delta_{n-1}), \tag{8.5}$$
$$H_{m,+}^w = H_{m,+} + \operatorname{diag}(b_w(m+1)\delta_{m+1}). \tag{8.6}$$

The Jacobi matrices $H_{0,n}$ and $H_{m,+}$ correspond to τ, hence, H_n^v is the Jacobi matrix corresponding to $\tau_n = \tau + b_v(n-1)\delta_{n-1}$, i.e.

$$H_n^v = \begin{pmatrix} b(1) & a(1) & & & \\ a(1) & \ddots & \ddots & & \\ & \ddots & b(n-2) & a(n-2) \\ & & a(n-2) & b(n-1) + \frac{a(n-1)v(n)}{v(n-1)} \end{pmatrix} \tag{8.7}$$

and $H_{m,+}^w$ is the Jacobi operator corresponding to $\tau_m = \tau + b_w(m+1)\delta_{m+1}$, that is

$$H_{m,+}^w = \begin{pmatrix} b(m+1) + \frac{a(m)w(m)}{w(m+1)} & a(m+1) & & \\ a(m+1) & b(m+2) & a(m+2) & \\ & a(m+2) & b(m+3) & \ddots \\ & & \ddots & \ddots \end{pmatrix}. \tag{8.8}$$

As v/w we'll always use a Weyl solution $\tilde{u}_{+/-}(z) \in \ell^2(\pm\mathbb{N})$ of a Jacobi difference equation $\tilde{\tau}\tilde{u} = z\tilde{u}$ where $\tilde{a} = a$ holds. In the special case where v and w are Weyl solutions of $\tau u = zu$ we abbreviate H_n^z and $H_{m,+}^z$. For notational convenience we then moreover abbreviate the corresponding index set as $\mathscr{J}_z$ since it will always be evident from the context if we use $\mathscr{J}_v$ or $\mathscr{J}_w$. Although $\tilde{u}_\pm(z)$ is only unique up to a multiple, the index set $\mathscr{J}_{\tilde{u}_\pm(z)}$ is unique and independent of the chosen multiple. Moreover, $\mathscr{J}_{\tilde{u}_\pm(z)}$ is an infinite set since $\tilde{u}_\pm(z)$ cannot have two consecutive zeros.

Whenever we add a boundary condition to a Jacobi operator, the corresponding spectral parameter z is in a gap of the essential spectrum and thus the Weyl solutions $u_\pm(z)$ always exist by Lemma 2.25.

We remark that we could also use the matrices

$$H_{m,0}^w = H_{m,0} + \operatorname{diag}(b_w(m+1)\delta_{m+1}), \tag{8.9}$$
$$H_{-,n}^v = H_{-,n} + \operatorname{diag}(b_v(n-1)\delta_{n-1}), \tag{8.10}$$

or even

$$H^{w,v}_{m,n} = H_{m,n} + \operatorname{diag}(b_w(m+1)\delta_{m+1}) + \operatorname{diag}(b_v(n-1)\delta_{n-1}) \tag{8.11}$$

as approximating sequences in the sequel. For notational convenience and to obtain our main theorem for H and H_+ we go up the half-line to H_+ and then back the other half-line to H.

Definition 8.1. *Let $\ell^2_0(\mathbb{N}_0)$ denote the linear spaces of all sequences with compact support equipped with the $\|.\|_2$-norm.*

The set $\ell^2_0(\mathbb{N}_0)$ is a dense meager set in the second category set $\ell^2(\mathbb{N}_0)$.

Lemma 8.2. *We have $\overline{\ell^2_0(\mathbb{N}_0)} = \ell^2(\mathbb{N}_0)$ and $\ell^2_0(\mathbb{N}_0)$ is a core of H_+.*

Proof. Clearly, $\ell^2_0(\mathbb{N}_0) \subseteq \ell^2(\mathbb{N}_0)$ holds. Since $\ell^2(\mathbb{N}_0)$ is a Hilbert space, and hence closed, we have $\overline{\ell^2_0(\mathbb{N}_0)} \subseteq \ell^2(\mathbb{N}_0)$. On the other hand, let $x \in \ell^2(\mathbb{N}_0)$ and let

$$x_n(m) = \begin{cases} x(m) & \text{if } m \leqslant n \\ 0 & \text{if } m > n, \end{cases} \tag{8.12}$$

then $x_n \in \ell^2_0(\mathbb{N}_0)$ for all n and $\lim_{n\to\infty}\|x_n - x\|_2 = 0$. Hence, every $x \in \ell^2(\mathbb{N}_0)$ is the limit of a sequence of elements of $\ell^2_0(\mathbb{N}_0)$ and thus $\ell^2(\mathbb{N}_0) \subseteq \overline{\ell^2_0(\mathbb{N}_0)}$. Since $\ell^2_0(\mathbb{N}_0)$ is a dense linear subspace of $\ell^2(\mathbb{N}_0)$ and H_+ is bounded, $\ell^2_0(\mathbb{N}_0)$ is a core of H_+, therefore confer Section 2.3. □

Analogously we define the space $\ell^2_0(\mathbb{Z})$, which is a core of H, and $\ell^2_0(-\mathbb{N}_0)$, which is a core of H_-.

Lemma 8.3. *Let $z_0 \in \mathbb{R}$. If $v \in \ell(\mathbb{N})$, then, as $n \to \infty, n \in \mathscr{J}_v$,*

$$H^v_n \oplus z_0\mathbb{I} \overset{sr}{\to} H_+ \quad \textit{and} \quad H^v_{-,n} \oplus z_0\mathbb{I} \overset{sr}{\to} H. \tag{8.13}$$

If $w \in \ell(-\mathbb{N})$, then, as $m \to -\infty, m \in \mathscr{J}_w$,

$$z_0\mathbb{I} \oplus H^w_{m,0} \overset{sr}{\to} H_- \quad \textit{and} \quad z_0\mathbb{I} \oplus H^w_{m,+} \overset{sr}{\to} H. \tag{8.14}$$

Proof. We only carry out the first claim: by Lemma 8.2 $\mathscr{D}_0 = \ell^2_0(\mathbb{N}_0)$ is a core of H_+. Moreover, for all $\psi \in \mathscr{D}_0$ there exists an $n_0(\psi) \in \mathbb{N}$ such that $\psi(j) = 0$ for all $j \geqslant n_0(\psi)$. Hence, $(H^v_n \oplus z_0\mathbb{I})\psi = H_+\psi$ for all $n > n_0(\psi) + 1, n \in \mathscr{J}_v$. Hence, $H^v_n \oplus z_0\mathbb{I} \overset{sr}{\to} H_+$ now follows from Theorem 2.21.a. □

In fact we even have strong convergence in the previous lemma, which (in the case of bounded operators) implies strong resolvent convergence, see Theorem 2.21. But the previous proof remains valid even if the operators are unbounded.

8.1.1 Open and half-open spectral intervals

From now on we assume

$$[z_-, z_+] \cap \sigma_{ess}(H) = \emptyset, \quad z_- < z_+. \tag{8.15}$$

By $\sigma_{ess}(H) = \sigma_{ess}(H_{-,m}) \cup \sigma_{ess}(H_{m,+})$ and since $H^w_{m,+}$ is a rank one perturbation of $H_{m,+}$, we then also have $[z_-, z_+] \cap \sigma_{ess}(H_{m,+}) = \emptyset$ for all $m \in \mathscr{J}_w$. Due to strong resolvent convergence (which we've shown to hold independently of the modified boundary condition) we easily obtain the following inequality on open spectral intervals.

Lemma 8.4. *If $v \in \ell(\mathbb{N})$, then*

$$\liminf_{\substack{n\to\infty \\ n\in\mathscr{J}_v}} \operatorname{tr}(P_{(z_-,z_+)}(H^v_n)) \geqslant \operatorname{tr}(P_{(z_-,z_+)}(H_+)). \tag{8.16}$$

If $w \in \ell(\mathbb{N})$, then

$$\liminf_{\substack{m\to-\infty \\ m\in\mathscr{J}_w}} \operatorname{tr}(P_{(z_-,z_+)}(H^w_{m,+})) \geqslant \operatorname{tr}(P_{(z_-,z_+)}(H)). \tag{8.17}$$

Proof. Let $z_0 \in \mathbb{R}, z_0 \notin [z_-, z_+]$, then by Theorem 2.9 $H^v_n \oplus z_0\mathbb{I}$ is self-adjoint and $\sigma(H^v_n \oplus z_0\mathbb{I}) = \sigma(H^v_n) \cup \{z_0\}$ holds. Thus, by Lemma 8.3 and Lemma 2.19 we have

$$\liminf_{\substack{n\to\infty \\ n\in\mathscr{J}_v}} \operatorname{tr}(P_{(z_-,z_+)}(H^v_n)) = \liminf_{\substack{n\to\infty \\ n\in\mathscr{J}_v}} \operatorname{tr}(P_{(z_-,z_+)}(H^v_n \oplus z_0\mathbb{I})) \geqslant \operatorname{tr}(P_{(z_-,z_+)}(H_+)).$$

The second claim can be obtained analogously. □

We notice that in some cases this is indeed a strict inequality, therefore consider the following example.

Remark 8.5. *Let $[z_-, z_+] \cap \sigma(H_+) = \emptyset$ and let v be a solution of $(\tau - z)v = 0$ such that $v(z, 0) = 0, z \in (z_-, z_+)$. Then, $z \in \sigma(H^v_n)$ for all $n \in \mathscr{J}_v$ and thus*

$$\liminf_{\substack{n\to\infty \\ n\in\mathscr{J}_v}} \operatorname{tr}(P_{(z_-,z_+)}(H^v_n)) > \operatorname{tr}(P_{(z_-,z_+)}(H_+)) = 0.$$

Recall the following lemma from functional analysis:

Lemma 8.6. *[42, Lemma 4.6]. Let $z_- < z_+, A \in \mathscr{L}(\mathscr{H})$ be self-adjoint on a (separable) Hilbert space $\mathscr{H}$. Let $\omega_j \in \mathscr{H}, 1 \leqslant j \leqslant k$, be linearly independent. If for any linear combination $\omega = \sum_{j=1}^k c_j\omega_j \neq 0$*

$$\|(A - \frac{z_+ + z_-}{2})\omega\| < \frac{z_+ - z_-}{2}\|\omega\| \tag{8.18}$$

holds, then $\dim \operatorname{Ran} P_{(z_-,z_+)}(A) \geqslant k$.

With the help of the previous lemma we'll now show that in all cases we're interested in (that is, we allow the boundary condition to come from a Weyl solution of a foreign operator which is not too far away) the previous inequalities cannot be strict.

Lemma 8.7. *Let* $v = \tilde{u}_+(\tilde{\lambda})$ *and* $w = \tilde{u}_-(\tilde{\lambda})$ *be Weyl solutions of* $(\tilde{\tau} - \tilde{\lambda})\tilde{u} = 0$. *If* $\tilde{\lambda} + b(j) - \tilde{b}(j) \in [z_-, z_+]$ *for all* $j \geqslant n$, $n \in \mathscr{J}_v$, *then*

$$\operatorname{tr}(P_{(z_-,z_+)}(H_n^v)) \leq \operatorname{tr}(P_{(z_-,z_+)}(H_+)). \tag{8.19}$$

If $\tilde{\lambda} + b(j) - \tilde{b}(j) \in [z_-, z_+]$ *for all* $j \leqslant m$, $m \in \mathscr{J}_w$, *then*

$$\operatorname{tr}(P_{(z_-,z_+)}(H_{m,+}^w)) \leq \operatorname{tr}(P_{(z_-,z_+)}(H)). \tag{8.20}$$

Proof. Let $n \in \mathscr{J}_v$ such that $\tilde{\lambda} + b(m) - \tilde{b}(m) \in [z_-, z_+]$ holds for all $m \geqslant n$ and let $e_1, \ldots, e_k$ be the eigenvalues of H_n^v in (z_-, z_+) with corresponding orthonormal eigenvectors $\vec{u}_1, \ldots, \vec{u}_k$, $k > 0$ (otherwise the claim holds obviously). To every eigenvector $\vec{u}_j, j = 1, \ldots k$, we choose a sequence $\omega_j \in \ell^2(\mathbb{N})$ such that

$$\omega_j(m) = \begin{cases} \vec{u}_j(m) & \text{if } 1 \leq m \leq n-1 \\ \gamma_j \tilde{u}_+(\tilde{\lambda}, m) & \text{if } m \geqslant n-1 \end{cases} \tag{8.21}$$

holds, where $\gamma_j \in \mathbb{R} \setminus \{0\}$ is chosen such that $\gamma_j \tilde{u}_+(\tilde{\lambda}, n-1) = \vec{u}_j(n-1)$ holds. We have $\tilde{u}_+(\tilde{\lambda}, n-1) \neq 0$ by $n \in \mathscr{J}_v$. The ω_j's are linearly independent elements of $\ell^2(\mathbb{N})$. Now, let $\psi = \sum_{j=1}^k c_j \omega_j \neq 0, c_j \in \mathbb{R}$. For all $m \geqslant n-1$ we have

$$\psi(m) = \sum_{j=1}^k c_j \gamma_j \tilde{u}_+(\tilde{\lambda}, m) = c\tilde{u}_+(\tilde{\lambda}, m),$$

where $c = \sum_{j=1}^k c_j \gamma_j$. Thus, for all $m \geqslant n$,

$$(H_+\psi)(m) = c(\tilde{H}_+ + b(m) - \tilde{b}(m))\tilde{u}_+(\tilde{\lambda}, m) = (\tilde{\lambda} + b(m) - \tilde{b}(m))\psi(m).$$

Hence, by $\tilde{\lambda} + b(m) - \tilde{b}(m) \in [z_-, z_+]$ for all $m \geqslant n$ we have

$$\sum_{m=n}^{\infty} |((H_+ - \frac{z_+ + z_-}{2})\psi)(m)|^2 = \sum_{m=n}^{\infty} |\tilde{\lambda} + b(m) - \tilde{b}(m) - \frac{z_+ + z_-}{2}|^2 |\psi(m)|^2$$
$$\leqslant (\frac{z_+ - z_-}{2})^2 \sum_{m=n}^{\infty} |\psi(m)|^2.$$

For all $j = 1, \ldots k$, we have

$$(H_+\omega_j)(n-1)$$

$$= a(n-1)\gamma_j \tilde{u}_+(\tilde{\lambda}, n) + a(n-2)\vec{u}_j(n-2) + b(n-1)\vec{u}_j(n-1)$$
$$= (H_n^v \vec{u}_j)(n-1) + a(n-1)\gamma_j \tilde{u}_+(\tilde{\lambda}, n) - \frac{a(n-1)\tilde{u}_+(\tilde{\lambda}, n)}{\tilde{u}_+(\tilde{\lambda}, n-1)}\vec{u}_j(n-1)$$
$$= e_j \vec{u}_j(n-1) + a(n-1)\gamma_j \tilde{u}_+(\tilde{\lambda}, n) - \frac{\gamma_j a(n-1)\tilde{u}_+(\tilde{\lambda}, n)}{\vec{u}_j(n-1)}\vec{u}_j(n-1)$$
$$= e_j \vec{u}_j(n-1)$$

and $(H_+\omega_j)(m) = (H_n^v \vec{u}_j)(m) = e_j \vec{u}_j(m)$ for all $m = 1, \ldots, n-2$.
Let $\vec{\psi} = \psi|_{\ell(0,n)}$, then $\langle \vec{u}_j, \vec{\psi}\rangle = c_j$. Let $\vec{\Phi} \in \operatorname{span}\{\vec{u}_1, \ldots, \vec{u}_k\}$ such that $\langle \vec{u}_j, \vec{\Phi}\rangle = (e_j - \frac{z_+ + z_-}{2})c_j$, then by Parseval's identity we have

$$\sum_{m=1}^{n-1} |((H_+ - \frac{z_+ + z_-}{2})\psi)(m)|^2 = \sum_{m=1}^{n-1} \left| \sum_{j=1}^{k} c_j ((H_+ - \frac{z_+ + z_-}{2})\omega_j)(m) \right|^2$$
$$= \sum_{m=1}^{n-1} \left| \sum_{j=1}^{k} (e_j - \frac{z_+ + z_-}{2}) c_j \omega_j(m) \right|^2 = \|\vec{\Phi}\|^2 = \sum_{j=1}^{k} |\langle \vec{u}_j, \vec{\Phi}\rangle|^2$$
$$= \sum_{j=1}^{k} |(e_j - \frac{z_+ + z_-}{2})|^2 |\langle \vec{u}_j, \vec{\psi}\rangle|^2$$
$$< (\frac{z_+ - z_-}{2})^2 \sum_{j=1}^{k} |\langle \vec{u}_j, \vec{\psi}\rangle|^2 = (\frac{z_+ - z_-}{2})^2 \|\vec{\psi}\|^2 = (\frac{z_+ - z_-}{2})^2 \sum_{m=1}^{n-1} |\psi(m)|^2$$

by $e_j \in (z_-, z_+)$. Now, the claim holds by Lemma 8.6 and

$$\|(H_+ - \frac{z_+ + z_-}{2})\psi\|^2$$
$$= \sum_{m=1}^{n-1} |((H_+ - \frac{z_+ + z_-}{2})\psi)(m)|^2 + \sum_{m=n}^{\infty} |((H_+ - \frac{z_+ + z_-}{2})\psi)(m)|^2$$
$$< (\frac{z_+ - z_-}{2})^2 \|\psi\|^2.$$

For the second claim let $e_1, \ldots, e_k$ be the eigenvalues of $H^w_{m,+}$ in (z_-, z_+) with corresponding orthonormal eigenvectors $\vec{u}_1, \ldots, \vec{u}_k$, $k > 0$. To every eigenvector $\vec{u}_j, j = 1, \ldots k$, we choose a sequence $\omega_j \in \ell^2(\mathbb{Z})$ such that

$$\omega_j(n) = \begin{cases} \vec{u}_j(n) & \text{if } m+1 \leq n \\ \gamma_j \tilde{u}_-(\tilde{\lambda}, n) & \text{if } n \leqslant m+1 \end{cases} \tag{8.22}$$

holds, where $\gamma_j \in \mathbb{R} \setminus \{0\}$ is chosen such that $\gamma_j \tilde{u}_-(\tilde{\lambda}, m+1) = \vec{u}_j(m+1)$.

Again, for all linear combinations ψ of the $\omega_j's$ we have

$$\sum_{n=-\infty}^{m} |((H - \frac{z_+ + z_-}{2})\psi)(n)|^2 \leqslant (\frac{z_+ - z_-}{2})^2 \sum_{n=-\infty}^{m} |\psi(n)|^2 \tag{8.23}$$

by $\tilde{\lambda} + b(n) - \tilde{b}(n) \in [z_-, z_+]$ for all $n \leqslant m$ and

$$\sum_{n=m+1}^{\infty} |((H - \frac{z_+ + z_-}{2})\psi)(n)|^2 < (\frac{z_+ - z_-}{2})^2 \sum_{n=m+1}^{\infty} |\psi(n)|^2. \tag{8.24}$$

□

Thus, of course we have equality now:

Lemma 8.8. *Let $v = \tilde{u}_+(\tilde{\lambda})$, $w = \tilde{u}_-(\tilde{\lambda})$ be Weyl solutions of $(\tilde{\tau} - \tilde{\lambda})\tilde{u} = 0$. If $\tilde{\lambda} + b(j) - \tilde{b}(j) \in [z_-, z_+]$ near ∞, then*

$$\lim_{\substack{n\to\infty \\ n\in \mathscr{J}_v}} \operatorname{tr}(P_{(z_-,z_+)}(H_n^v)) = \operatorname{tr}(P_{(z_-,z_+)}(H_+)). \tag{8.25}$$

If $\tilde{\lambda} + b(j) - \tilde{b}(j) \in [z_-, z_+]$ near $-\infty$, then

$$\lim_{\substack{m\to-\infty \\ m\in \mathscr{J}_w}} \operatorname{tr}(P_{(z_-,z_+)}(H_{m,+}^w)) = \operatorname{tr}(P_{(z_-,z_+)}(H)). \tag{8.26}$$

Proof. By Lemma 8.7 $\limsup_{\substack{n\to\infty \\ n\in \mathscr{J}_v}} \operatorname{tr}(P_{(z_-,z_+)}(H_n^v)) \leqslant \operatorname{tr}(P_{(z_-,z_+)}(H_+))$ holds and by Lemma 8.4 we have $\liminf_{\substack{n\to\infty \\ n\in \mathscr{J}_v}} \operatorname{tr}(P_{(z_-,z_+)}(H_n^v)) \geqslant \operatorname{tr}(P_{(z_-,z_+)}(H_+))$. Thus, the limit exists and the first claim holds. The second claim can be obtained analogously. □

We point out the following special case to which we'll frequently refer in the sequel.

Corollary 8.9. *Let $u_\pm(\lambda)$ be Weyl solutions of $(\tau - \lambda)u = 0$, $\lambda \in [z_-, z_+]$, then*

$$\lim_{\substack{n\to\infty \\ n\in \mathscr{J}_\lambda}} \operatorname{tr}(P_{(z_-,z_+)}(H_n^\lambda)) = \operatorname{tr}(P_{(z_-,z_+)}(H_+)), \tag{8.27}$$

and

$$\lim_{\substack{m\to-\infty \\ m\in \mathscr{J}_\lambda}} \operatorname{tr}(P_{(z_-,z_+)}(H_{m,+}^\lambda)) = \operatorname{tr}(P_{(z_-,z_+)}(H)). \tag{8.28}$$

It can easily be seen that under certain assumptions we even obtain equality at half-open spectral intervals, a very helpful lemma for our subsequent investigations.

Lemma 8.10. *Let $v = \tilde{u}_+(\tilde{\lambda})$, $w = \tilde{u}_-(\tilde{\lambda})$ be Weyl solutions of $(\tilde{\tau} - \tilde{\lambda})\tilde{u} = 0$. Then,*

$$\lim_{\substack{n\to\infty \\ n\in\mathscr{J}_v}} \operatorname{tr}(P_{(z_-,z_+]}(H_n^v)) = \operatorname{tr}(P_{(z_-,z_+]}(H_+)), \tag{8.29}$$

$$\lim_{\substack{m\to-\infty \\ m\in\mathscr{J}_w}} \operatorname{tr}(P_{(z_-,z_+]}(H_{m,+}^v)) = \operatorname{tr}(P_{(z_-,z_+]}(H)) \tag{8.30}$$

holds if $\tilde{\lambda} + b(j) - \tilde{b}(j) \downarrow z_+$ as $j \to \pm\infty$ and

$$\lim_{\substack{n\to\infty \\ n\in\mathscr{J}_v}} \operatorname{tr}(P_{[z_-,z_+)}(H_n^v)) = \operatorname{tr}(P_{[z_-,z_+)}(H_+)), \tag{8.31}$$

$$\lim_{\substack{m\to-\infty \\ m\in\mathscr{J}_w}} \operatorname{tr}(P_{[z_-,z_+)}(H_{m,+}^v)) = \operatorname{tr}(P_{[z_-,z_+)}(H)) \tag{8.32}$$

holds if $\tilde{\lambda} + b(j) - \tilde{b}(j) \uparrow z_-$ as $j \to \pm\infty$.

Proof. Let $\varepsilon > 0$ be sufficiently small such that $[z_- - \varepsilon, z_+ + \varepsilon] \cap \sigma_{ess}(H_+) = \emptyset$. Suppose $\lim_{j\to\infty} \tilde{\lambda} + b(j) - \tilde{b}(j) \downarrow z_+$, then $\tilde{\lambda} + b(j) - \tilde{b}(j) \in [z_+, z_+ + \varepsilon]$ near ∞ and hence by Lemma 8.8 we have

$$\lim_{\substack{n\to\infty \\ n\in\mathscr{J}_v}} \operatorname{tr}(P_{(z_-,z_++\varepsilon)}(H_n^v)) = \operatorname{tr}(P_{(z_-,z_++\varepsilon)}(H_+)),$$

$$\lim_{\substack{n\to\infty \\ n\in\mathscr{J}_v}} \operatorname{tr}(P_{(z_+,z_++\varepsilon)}(H_n^v)) = \operatorname{tr}(P_{(z_+,z_++\varepsilon)}(H_+)).$$

The same holds for $H_{m,+}^v$. For the second claim use $\tilde{\lambda} + b(j) - \tilde{b}(j) \in [z_- - \varepsilon, z_-]$ near $\pm\infty$ and Lemma 8.8 again. □

The following is an important special case, also for half-open intervals.

Corollary 8.11. *We have*

$$\lim_{\substack{n\to\infty \\ n\in\mathscr{J}_v}} \operatorname{tr}(P_{(z_-,z_+]}(H_n^{z_+})) = \operatorname{tr}(P_{(z_-,z_+]}(H_+)), \tag{8.33}$$

$$\lim_{\substack{m\to-\infty \\ m\in\mathscr{J}_w}} \operatorname{tr}(P_{(z_-,z_+]}(H_{m,+}^{z_+})) = \operatorname{tr}(P_{(z_-,z_+]}(H)), \tag{8.34}$$

and

$$\lim_{\substack{n\to\infty \\ n\in\mathscr{J}_v}} \operatorname{tr}(P_{[z_-,z_+)}(H_n^{z_-})) = \operatorname{tr}(P_{[z_-,z_+)}(H_+)), \tag{8.35}$$

$$\lim_{\substack{m\to-\infty \\ n\in\mathscr{J}_w}} \operatorname{tr}(P_{[z_-,z_+)}(H_{m,+}^{z_-})) = \operatorname{tr}(P_{[z_-,z_+)}(H)). \tag{8.36}$$

8.1.2 A point

We discuss whether or not a point is an eigenvalue of the approximating matrix (with a modified boundary condition coming from v/w) and how this question is answered by the Wronskian.

Lemma 8.12. *Let v be a solution of $(\tilde{\tau} - \tilde{\lambda})v = 0$ and $n \in \mathscr{J}_v$. Let $m \in \mathbb{Z}$, then,*

$$\lambda \in \sigma(H^v_{m,n}) \quad \Longleftrightarrow \quad W_{n-1}(\psi_m(\lambda), v) = 0, \tag{8.37}$$

where $\psi_m(\lambda)$ denotes a solution of $(\tau - \lambda)\psi_m(\lambda) = 0$ such that $\psi_m(\lambda, m) = 0$. Moreover,

$$\lambda \in \sigma(H^v_{-,n}) \quad \Longleftrightarrow \quad W_{n-1}(u_-(\lambda), v) = 0, \tag{8.38}$$

where $u_-(\lambda)$ denotes a solution of $(\tau - \lambda)u_-(\lambda) = 0$ which is square summable near $-\infty$.

Proof. Since the difference equations τ and τ_n coincide below $b(n-1)$ there exists a solution $\psi(\lambda)$ of $(\tau_n - \lambda)\psi = 0$ such that $\psi_m(\lambda, j) = \psi(\lambda, j)$ for all $m \leqslant j < n$. Moreover, $\psi(\lambda, n) = 0 \iff \lambda \in \sigma(H^v_n)$. We have

$$\begin{aligned} &- a(n-1)\psi(n) \\ &\quad = a(n-2)\psi(n-2) + (b(n-1) + \frac{a(n-1)v(n)}{v(n-1)} - \lambda)\psi(n-1) \\ &\quad = -a(n-1)\psi_m(n) + \frac{a(n-1)v(n)}{v(n-1)}\psi_m(n-1), \end{aligned}$$

thus,

$$-a(n-1)\psi(n)v(n-1) = W_{n-1}(\psi_m(\lambda), v).$$

For the second claim let $\psi(\lambda)$ be a solution of $(\tau_n - \lambda)\psi = 0$ such that $u_-(\lambda, j) = \psi(\lambda, j)$ for all $j < n$. □

Lemma 8.13. *Let w be a solution of $(\tilde{\tau} - \tilde{\lambda})w = 0$ and $m \in \mathscr{J}_w$. Let $n \in \mathbb{Z}$, then,*

$$\lambda \in \sigma(H^w_{m,n}) \quad \Longleftrightarrow \quad W_m(w, \psi_n(\lambda)) = 0, \tag{8.39}$$

where $\psi_n(\lambda)$ denotes a solution of $(\tau - \lambda)\psi_n(\lambda) = 0$ such that $\psi_n(\lambda, n) = 0$. Moreover,

$$\lambda \in \sigma(H^w_{m,+}) \quad \Longleftrightarrow \quad W_m(w, u_+(\lambda)) = 0, \tag{8.40}$$

where $u_+(\lambda)$ denotes a solution of $(\tau - \lambda)u_+(\lambda) = 0$ which is square summable near $-\infty$.

Proof. Since the difference equations τ and τ_m coincide abow $b(m+1)$ there exists a solution $\psi(\lambda)$ of $(\tau_m - \lambda)\psi = 0$ such that $u_n(\lambda, j) = \psi(\lambda, j)$ for all $m < j \leqslant n$. Moreover, $\psi(\lambda, m) = 0 \iff \lambda \in \sigma(H^w_{m,n})$. We have

$$\begin{aligned} -a(m)\psi(m) &= a(m+1)\psi(m+2) + (b(m+1) + \frac{a(m)w(m)}{w(m+1)} - \lambda)\psi(m+1) \\ &= -a(m)\psi_n(m) + \frac{a(m)w(m)}{w(m+1)}\psi_n(m+1), \end{aligned}$$

thus,

$$-a(m)\psi(m)w(m+1) = W_m(w, \psi_n(\lambda)).$$

For the second claim let $\psi(\lambda)$ be a solution of $(\tau_m - \lambda)\psi = 0$ such that $u_+(\lambda, j) = \psi(\lambda, j)$ for all $m < j$. □

This leads us again to the following important special cases which should be mentioned separately.

Corollary 8.14. *We have*

$$\lambda \in \sigma(H_+) \iff \lambda \in \sigma(H_n^\lambda) \text{ for one (and hence for all) } n \in \mathscr{J}_\lambda,$$
$$\lambda \in \sigma(H) \iff \lambda \in \sigma(H_{m,+}^\lambda) \text{ for one (and hence for all) } m \in \mathscr{J}_\lambda.$$

Proof. By Lemma 8.12

$$\lambda \in \sigma(H_+) \iff W(\psi_0(\lambda), u_+(\lambda)) \text{ vanishes}$$

holds, respectively by Lemma 8.13 we have

$$\lambda \in \sigma(H) \iff W(u_-(\lambda), u_+(\lambda)) \text{ vanishes.}$$

□

Clearly, Corollary 8.9 and Corollary 8.14 also imply Corollary 8.11.

Corollary 8.15. *Let v be a solution of $(\tilde\tau - \lambda)v = 0$ and $n \in \mathscr{J}_v$. Let $\lambda \in \sigma_d(H_+)$, then,*

$$\lambda \in \sigma(H_n^v) \iff W_{n-1}(u_+(\lambda), v) = 0,$$

where $u_+(\lambda)$ is the corresponding eigensequence of H_+.

In particular, it can now easily be seen that a point is at most finitely many times in the spectrum of the approximating matrices if the boundary condition comes from a Weyl solution corresponding to some *foreign* spectral parameter.

Lemma 8.16. *Let $b \downarrow \tilde b$ or $b \uparrow \tilde b$, $\lambda, \tilde\lambda \notin \sigma_{ess}(H_+)$, and $\lambda \neq \tilde\lambda$.*
Fix some $m \in \mathbb{Z}$. If $\tau - \lambda \overset{rno_+}{\sim} \tilde\tau - \tilde\lambda$ and v is a solution of $(\tilde\tau - \tilde\lambda)v = 0$, then

$$\lambda \notin \sigma(H_{-,n}^v) \quad \text{and} \quad \lambda \notin \sigma(H_{m,n}^v) \tag{8.41}$$

for all $n \in \mathscr{J}_v$ sufficiently large.
Fix some $n \in \mathbb{Z}$. If $\tau - \lambda \overset{rno_-}{\sim} \tilde\tau - \tilde\lambda$ and w is a solution of $(\tilde\tau - \tilde\lambda)w = 0$, then

$$\lambda \notin \sigma(H_{m,+}^w) \quad \text{and} \quad \lambda \notin \sigma(H_{m,n}^w) \tag{8.42}$$

for all $m \in \mathscr{J}_w$, $|m|$ sufficiently large.

Proof. By Lemma 7.6 and $\lambda \neq \tilde{\lambda}$ we have $W(u_-(\lambda), v) \neq 0$, $W(\psi_m(\lambda), v) \neq 0$ near $+\infty$ and $W(w, u_+) \neq 0$, $W(w, \psi_n) \neq 0$ near $-\infty$. Now use Lemma 8.12 and Lemma 8.13. □

8.2 . . . of the Wronskian with suitable boundary conditions

We approximate a Wronskian which consists of solutions fulfilling the left/right boundary condition of two (different) Jacobi operators on the half line (and in the second step on the line) with Wronskians of solutions fulfilling the left/right boundary condition of the two (different) approximating problems and compare their number of nodes. In contrary to the next section we assume that one of the approximated solutions generates the boundary conditions of the approximating problems.

We'll later reuse the notation introduced here, thus, for the convinience of the reader, we split our considerations in two parts: the half-line and the line.

8.2.1 Nodes on the half-line

Consider the following setting: let

$$H_n^v \oplus z_0 \mathbb{I} \stackrel{sr}{\to} H_+ \quad \text{and} \quad \tilde{H}_n^v \oplus z_0 \mathbb{I} \stackrel{sr}{\to} \tilde{H}_+,$$

$n \in \mathscr{J}_v$, where the boundary conditions of the approximating matrices are generated by a Weyl solution v corresponding to one of the two semi-infinite Jacobi operators, namely to H_+. Recall from (8.7) that

$$\tau_n = \tau + b_v(n-1)\delta_{n-1} \quad \text{and} \quad \tilde{\tau}_n = \tilde{\tau} + b_v(n-1)\delta_{n-1}$$

are the difference equations corresponding to the approximating matrices. Let $\psi_{n,j}(\lambda)$ be a solution of $(\tau_n - \lambda)\psi = 0$ such that $\psi_{n,j}(\lambda, j) = 0$.

Lemma 8.17. *Let $b \downarrow \tilde{b}$ or $b \uparrow \tilde{b}$ near ∞, $\tau - \lambda \stackrel{rno_+}{\sim} \tilde{\tau} - \tilde{\lambda}$, and $v = u_+(\lambda)$, then*

$$\lim_{n\to\infty} \#_{[0,n]}(\tilde{\psi}_{n,0}(\tilde{\lambda}), \psi_{n,n}(\lambda)) = \#_{[0,\infty]}(\tilde{u}_-(\tilde{\lambda}), u_+(\lambda)),$$
$$\lim_{n\to\infty} \#_{[0,n]}(\psi_{n,n}(\lambda), \tilde{\psi}_{n,0}(\tilde{\lambda})) = \#_{[0,\infty]}(u_+(\lambda), \tilde{u}_-(\tilde{\lambda})),$$

where u_- denotes a solution fulfilling the left Dirichlet boundary condition of H_+, that is, $u_-(0) = 0$.

The same holds if we replace $\#_{[0,\cdot]}$ on both sides by $\#_{[0,\cdot)}$, $\#_{(0,\cdot]}$, or $\#_{(0,\cdot)}$.

Proof. Let n such that by Lemma 7.4 the Wronskian $W(\tilde{u}_-(\tilde{\lambda}), u_+(\lambda))$ is of one sign above $n-1$. Since the difference equations τ and τ_n coincide below $n-1$

the solutions $\tilde\psi_{n,0}(\tilde\lambda)$ and $\tilde u_-(\tilde\lambda)$ also coincide (up to a multiple) below $n-1$. The same holds for $\psi_{n,n}(\lambda)$ and $u_+(\lambda)$ by $v = u_+(\lambda)$. Thus, without loss (pick suitable multiples),

$$W_m(\tilde\psi_{n,0}(\tilde\lambda), \psi_{n,n}(\lambda)) = W_m(\tilde u_-(\tilde\lambda), u_+(\lambda))$$

holds at $m = 0, \dots, n-2$ and moreover at $m = n-1$ by $\tilde b^n - b^n = \tilde b - b$ and

$$\begin{aligned}
&W_{n-1}(\tilde\psi_{n,0}(\tilde\lambda), \psi_{n,n}(\lambda))\\
&= W_{n-2}(\tilde\psi_{n,0}(\tilde\lambda), \psi_{n,n}(\lambda))\\
&\quad + ((\tilde b^n - b^n)(n-1) + \lambda - \tilde\lambda)\tilde\psi_{n,0}(n-1), \psi_{n,n}(n-1)\\
&= W_{n-2}(\tilde u_-, u_+) + ((\tilde b - b)(n-1) + \lambda - \tilde\lambda)\tilde u_-(n-1)u_+(n-1)\\
&= W_{n-1}(\tilde u_-, u_+).
\end{aligned}$$

We have $W_n(\tilde\psi_{n,0}(\tilde\lambda), \psi_{n,n}(\lambda)) = W_{n-1}(\tilde\psi_{n,0}(\tilde\lambda), \psi_{n,n}(\lambda))$ by $\psi_{n,n}(n) = 0$, hence the first claim now holds by $\#_{n-1}(\tilde u_-(\tilde\lambda), u_+(\lambda)) = 0$ and

$$W_n(\tilde\psi_{n,0}(\tilde\lambda), \psi_{n,n}(\lambda)) = W_{n-1}(\tilde\psi_{n,0}(\tilde\lambda), \psi_{n,n}(\lambda)) = W_{n-1}(\tilde u_-(\tilde\lambda), u_+(\lambda)).$$

The second claim holds analogously. □

Hence, we have seen that the Wronskians corresponding to the approximating finite problems in the limit have equally many nodes as the Wronskian corresponding to the semi-infinite operators. This comes from the fact that the boundary conditions have been generated carefully.
The following corollary states that for Wronskians of solutions corresponding to different spectral parameters we can slightly ease the counting method since we already know that in this case the Wronskian cannot vanish near ∞.

Corollary 8.18. *Let $b \downarrow \tilde b$ or $b \uparrow \tilde b$ near ∞, $\tau - \lambda \overset{rno_+}{\sim} \tilde\tau - \tilde\lambda$, $\lambda \neq \tilde\lambda$, and $v = u_+(\lambda)$, then*

$$\begin{aligned}
\lim_{n\to\infty} \#_{[0,n)}(\tilde\psi_{n,0}(\tilde\lambda), \psi_{n,n}(\lambda)) &= \#_{[0,\infty]}(\tilde u_-(\tilde\lambda), u_+(\lambda)),\\
\lim_{n\to\infty} \#_{[0,n)}(\psi_{n,n}(\lambda), \tilde\psi_{n,0}(\tilde\lambda)) &= \#_{[0,\infty]}(u_+(\lambda), \tilde u_-(\tilde\lambda))
\end{aligned}$$

where u_- denotes a solution fulfilling the left Dirichlet boundary condition of H_+, that is, $u_-(0) = 0$, and the Wronskians don't vanish near $+\infty$.
The same holds if we replace $\#_{[0,n)}$ by $\#_{(0,n)}$ and $\#_{[0,\infty]}$ by $\#_{(0,\infty]}$.

Proof. Use Lemma 8.17 and Lemma 7.6. □

And of course we now already get a first equality between the spectra of (this specific sequence of) *finite* matrices and the Wronskian on the *half-line*.

Lemma 8.19. *Let $b \downarrow \tilde{b}$ or $b \uparrow \tilde{b}$ near ∞, $\tau - \lambda \overset{rno_+}{\sim} \tilde{\tau} - \tilde{\lambda}$, and $v = u_+(\lambda)$, then there exist sequences of approximating matrices (which depend on the spectral parameter) such that*

$$E_{(-\infty,\tilde{\lambda})}(\tilde{H}_n^v) - E_{(-\infty,\lambda]}(H_n^\lambda) = \#_{(0,\infty]}(u_+(\lambda), \tilde{u}_-(\tilde{\lambda})), \tag{8.43}$$

$$E_{(-\infty,\lambda)}(H_n^\lambda) - E_{(-\infty,\tilde{\lambda}]}(\tilde{H}_n^{0,v}) = \#_{(0,\infty]}(\tilde{u}_-(\tilde{\lambda}), u_+(\lambda)) \tag{8.44}$$

for all $n \in \mathscr{J}_v$ sufficiently large, where u_- denotes a solution fulfilling the left Dirichlet boundary condition of H_+, that is, $u_-(0) = 0$.

Proof. By $\lambda \notin \sigma_{ess}(H_+^0)$ the Weyl solution and hence also the approximating matrices exist. For the first claim use

$$E_{(-\infty,\tilde{\lambda})}(\tilde{H}_n^v) - E_{(-\infty,\lambda]}(H_n^\lambda) = \#_{(0,n]}(\psi_{n,n}(\lambda), \tilde{\psi}_{n,0}(\tilde{\lambda}))$$

from Theorem 1.5 and Lemma 8.17. For the second claim use

$$E_{(-\infty,\lambda)}(H_n^\lambda) - E_{(-\infty,\tilde{\lambda}]}(\tilde{H}_n^v) = \#_{(0,n]}(\tilde{\psi}_{n,0}(\tilde{\lambda}), \psi_{n,n}(\lambda)).$$

□

8.2.2 Nodes on the line

Let

$$z_0\mathbb{I} \oplus H_{m,+}^w \overset{sr}{\to} H \quad \text{and} \quad z_0\mathbb{I} \oplus \tilde{H}_{m,+}^w \overset{sr}{\to} \tilde{H},$$

$m \in \mathscr{J}_w$, where the boundary conditions of the semi-infinite Jacobi operators are generated by a Weyl solution w corresponding to the infinite Jacobi operator H. Recall from (8.8) that

$$\tau_{j,m} = \tau_j + b_w(m+1)\delta_{m+1} \quad \text{and} \quad \tilde{\tau}_{j,m} = \tilde{\tau}_j + b_w(m+1)\delta_{m+1}$$

are the difference equations corresponding to the semi-infinite operators and let $\psi_{m,m}(\lambda)$ and $\psi_{m,+}(\lambda)$ be solutions of $(\tau_m - \lambda)\psi = 0$ such that

$$\psi_{m,m}(\lambda, m) = 0 \quad \text{and} \quad \psi_{m,+}(\lambda) \in \ell^2(\mathbb{N}).$$

In a similar manner we now show that, from some point on, each of the (suitably choosen) Wronskians corresponding to the approximating problems on the half-line has equally many nodes at $[m, \infty]$ as the Wronskian of two Weyl solutions.

Lemma 8.20. *Let $b \downarrow \tilde{b}$ or $b \uparrow \tilde{b}$ near $+\infty$ and near $-\infty$, $\tau - \lambda \overset{rno}{\sim} \tilde{\tau} - \tilde{\lambda}$, and $w = u_-(\lambda)$, then*

$$\lim_{m \to -\infty} \#_{[m,\infty]}(\psi_{m,m}(\lambda), \tilde{\psi}_{m,+}(\tilde{\lambda})) = \#_{[-\infty,\infty]}(u_-(\lambda), \tilde{u}_+(\tilde{\lambda})),$$

$$\lim_{m\to-\infty} \#_{[m,\infty]}(\tilde{\psi}_{m,+}(\tilde{\lambda}), \psi_{m,m}(\lambda)) = \#_{[-\infty,\infty]}(\tilde{u}_+(\tilde{\lambda}), u_-(\lambda)),$$

where u_- denotes a solution fulfilling $u_- \in \ell^2(-\mathbb{N})$. The same holds if we replace $\#_{[0,\cdot]}$ on both sides by $\#_{[0,\cdot)}$, $\#_{(0,\cdot]}$, or $\#_{(0,n)}$.

Proof. Let m such that by Lemma 7.4 the Wronskian $W(u_-(\lambda), \tilde{u}_+(\lambda))$ is of one sign below $m+1$. Since the difference equations τ and τ_m coincide above $m+2$ the solutions $\tilde{\psi}_{m,+}(\tilde{\lambda})$ and $\tilde{u}_+(\tilde{\lambda})$ also coincide (up to a multiple) above $m+1$. Moreover, a solution ψ of $(\tau_m - \lambda)\psi = 0$ which coincides with $u_-(\lambda)$ above m is a multiple of $\psi_{m,m}(\lambda)$ by

$$\begin{aligned}
&- a(m)\psi(m)\\
&\quad = a(m+1)\psi(m+2) + (b(m+1) - \lambda_0 + \frac{a(m)w(m)}{w(m+1)})\psi(m+1)\\
&\qquad\quad = -a(m)u_-(m) + \frac{a(m)u_-(m)}{u_-(m+1)}u_-(m+1) = 0. \qquad (8.45)
\end{aligned}$$

Thus, without loss (pick suitable multiples),

$$W_j(\psi_{m,m}(\lambda), \tilde{\psi}_{m,+}(\tilde{\lambda})) = W_m(u_-(\lambda), \tilde{u}_+(\tilde{\lambda}))$$

holds at $j \geqslant m+1$ and moreover at $j = m$ by

$$\begin{aligned}
&W_{m+1}(u_-(\lambda), \tilde{u}_+(\tilde{\lambda})) - W_m(\psi_{m,m}(\lambda), \tilde{\psi}_{m,+}(\tilde{\lambda}))\\
&\quad = W_{m+1}(\psi_{m,m}(\lambda), \tilde{\psi}_{m,+}(\tilde{\lambda})) - W_m(\psi_{m,m}(\lambda), \tilde{\psi}_{m,+}(\tilde{\lambda}))\\
&\quad = (b^n(m+1) - \lambda - \tilde{b}^n(m+1) + \tilde{\lambda})\psi_{m,m}(m+1)\tilde{\psi}_{m,+}(m+1)\\
&\quad = (b(m+1) - \lambda - \tilde{b}(m+1) + \tilde{\lambda})u_-(m+1)\tilde{u}_+(m+1)\\
&\quad = W_{m+1}(u_-(\lambda), \tilde{u}_+(\tilde{\lambda})) - W_m(u_-(\lambda), \tilde{u}_+(\tilde{\lambda})).
\end{aligned}$$

Thus,

$$\#_{[m,\infty]}(\psi_{m,m}(\lambda), \tilde{\psi}_{m,+}(\tilde{\lambda})) = \#_{[m,\infty]}(u_-(\lambda), \tilde{u}_+(\tilde{\lambda}))$$

and $\tau - \lambda \overset{rno}{\sim} \tilde{\tau} - \tilde{\lambda}$ proves the first claim. The second claim can be shown analogously. □

8.3 . . . of the Wronskian with foreign boundary conditions

The previous considerations will be enough to establish the relative oscillation theorems below the essential spectrum. But for a proof of our main theorem in gaps of the essential spectrum we further have to investigate the approximative behaviour of the Wronskian of two solutions (of two different equations) where

the Weyl solution v/w which generates the boundary conditions comes from (one of the operators but) some *foreign* spectral parameter. This section can be skipped for the proofs below the essential spectra in Chapter 9.
Recall that

$$\tau_n = \tau + b_v(n-1)\delta_{n-1}, \quad n \in \mathscr{J}_v,$$

and

$$\tau_m = \tau + b_w(m+1)\delta_{m+1}, \quad m \in \mathscr{J}_w$$

are the difference equations from (8.7) and (8.8).
In the first step we show that the solutions $\varphi_n(z)$ corresponding to the finite problems approximate the Weyl solution $u_+(z)$ at *finite sets* due to the convergence of the Weyl m-functions. Therefore, of course, we have to ensure that the Weyl m-functions exist, which follows from our previous considerations, see Lemma 8.16.

Lemma 8.21. *Let $v = \tilde{u}_+(\tilde{z})$, $\tilde{z} \neq z$, $\tau - z \overset{rno_+}{\sim} \tilde{\tau} - \tilde{z}$, $I \subset \mathbb{Z}$ be a finite set and let $u_+(z) \in \ell^2(\mathbb{N})$ be a Weyl solution of $(\tau - z)u = 0$. Then, for all $n \in \mathscr{J}_v$ there exists a solution $\varphi_n(z)$ of $(\tau_n - z)\varphi_n(z) = 0$ such that $\varphi_n(n) = 0$ and*

$$\lim_{\substack{n\to\infty \\ n\in\mathscr{J}_v}} \varphi_n(z,j) = u_+(z,j) \qquad \text{for all } j \in I. \tag{8.46}$$

Proof. Let $m < \min I$ such that $u_+(z,m) \neq 0$, let $H_{m,+}$ be a Jacobi operator corresponding to τ and let $H^v_{m,n}$ be Jacobi matrices corresponding to τ_n. Then, for some $\lambda \neq z$ we have $H^v_{m,n} \oplus \lambda\mathbb{I} \overset{sr}{\to} H_{m,+}$, $z \in \rho(H_{m,+})$, and $z \in \rho(H^v_{m,n})$ for all n sufficiently large by Lemma 8.16 and $\tilde{z} \neq z$. Hence, for all n sufficiently large, the corresponding Weyl m-functions exist and $m^n_+(z,m) \to m_+(z,m)$ as $n \to \infty$ by Lemma 2.32.
W.l.o.g. let $m = 0$ and let $c(z), s(z)$ denote a fundamental system of $\tau - z$ such that $c(z,0) = 1, c(z,1) = 0$ and $s(z,0) = 0, s(z,1) = 1$. Then, $u_+(z)$ is a linear combination of $c(z), s(z)$ and hence we have

$$\begin{aligned} u_+(j) &= u_+(0)c(j) + u_+(1)s(j) = a(0)u_+(0)(a(0)^{-1}c(j) + \frac{u_+(1)}{a(0)u_+(0)}s(j)) \\ &= a(0)u_+(0)(a(0)^{-1}c(j) - m_+(z,0)s(j)) \end{aligned}$$

for all $j \in \mathbb{Z}$ by $m_+(z,0) = \langle \delta_1, (H_{0,+} - z)^{-1}\delta_1 \rangle = -\frac{u_+(z,1)}{a(0)u_+(z,0)}$. Now, let $\phi_n(z)$ denote a solution of $(\tau_n - z)\phi_n(z) = 0$ such that $\phi_n(z,n) = 0$ and let $c_n(z), s_n(z)$ denote a fundamental system of $\tau_n - z$ such that $c_n(z,0) = 1, c_n(z,1) = 0$ and $s_n(z,0) = 0, s_n(z,1) = 1$. Then, $\phi_n(z)$ is a linear combination of $c_n(z), s_n(z)$ and hence we have

$$\phi_n(j) = \phi_n(0)c_n(j) + \phi_n(1)s_n(j) = a(0)\phi_n(0)(a(0)^{-1}c_n(j) + \frac{\phi_n(1)}{a(0)\phi_n(0)}s_n(j))$$

$$= a(0)\phi_n(0)(a(0)^{-1}c_n(j) - m_+^n(z,0)s_n(j))$$

by $m_+^n(z,0) = \langle\delta_1, (H_{0,n}^v - z)^{-1}\delta_1\rangle = -\frac{\phi_n(z,1)}{a(0)\phi_n(z,0)}$. The difference equations τ_n and τ coincide at I for all n sufficiently large and hence we have

$$\varphi_n(j) = a(0)\varphi_n(0)(a(0)^{-1}c(j) - m_+^n(z,0)s(j)) \quad \text{for all } j \in I.$$

By $\phi_n(0) \neq 0$ and $u_+(0) \neq 0$ there exists an $\alpha_n \neq 0$ such that $\varphi_n(z) = \alpha_n\phi_n(z)$ coincides with $u_+(z)$ at the point 0. Thus, for all $j \in I$ by

$$(\varphi_n - u_+)(j) = \alpha_n\phi_n(j) - u_+(j) = a(0)u_+(0)s(j)(m_+(z,0) - m_+^n(z,0)) \quad (8.47)$$

and $\lim_{\substack{n\to\infty \\ n\in\mathscr{J}_v}} m_+^n(z,0) = m_+(z,0)$ we have $\lim_{\substack{n\to\infty \\ n\in\mathscr{J}_v}} \varphi_n(j) = u_+(j)$. □

Remark 8.22. *Let $H_{0,n}^v \oplus z_0\mathbb{I} \overset{sr}{\to} H_{0,+}$, where v is a boundary condition corresponding to some spectral parameter $\tilde{z} \neq z$ and let $\varphi_n(z)$ be the solutions from the previous lemma such that $\varphi_n(z) \to u_+(z)$ at a finite set I, which contains the point 0. Then, by Lemma 8.16 we have $z \notin \sigma(H_{0,n})$, that is $\varphi_n(z,0) \neq 0$, for all $n \in \mathscr{J}_v$ sufficiently large, although we could have $u_+(z,0) = 0$, i.e. $z \in \sigma(H_{0,+})$. If so, then $W_0(u_+(z), \tilde{u}_-(z)) = 0$ and $W_0(\varphi_n(z), \tilde{u}_-(z)) \neq 0$ as $n \to \infty$.*
Hence, it can happen that, for all $n \in \mathscr{J}_v$ sufficiently large, the approximating Wronskians have one node more/less (depending on the counting method) than $W(u_+(z), \tilde{u}_-(z))$. Confer also Remark 10.4, Lemma 10.16, and Lemma 10.17.

Obviously, the same can de done in the other direction:

Lemma 8.23. *Let $w = \tilde{u}_-(\tilde{z})$, $\tilde{z} \neq z$, $\tau - z \overset{rno-}{\sim} \tilde{\tau} - \tilde{z}$, $I \subset \mathbb{Z}$ be a finite set and let $u_-(z) \in \ell^2(-\mathbb{N})$ be a Weyl solution of $(\tau - z)u = 0$. Then, for all $m \in \mathscr{J}_w$ there exists a solution $\varphi_m(z)$ of $(\tau_m - z)\varphi_m(z) = 0$ such that $\varphi_m(m) = 0$ and*

$$\lim_{\substack{m\to-\infty \\ m\in\mathscr{J}_w}} \varphi_m(z,j) = u_-(z,j) \qquad \textit{for all } j \in I.$$

Proof. Let $n > \max I$ such that $u_-(z,n) \neq 0$, let $H_{-,n}$ be a Jacobi operator corresponding to τ and let $H_{m,n}^w$ be Jacobi matrices corresponding to τ_m. Then, for some $\lambda \neq z$ we have $\lambda\mathbb{I} \oplus H_{m,n}^w \overset{sr}{\to} H_{-,n}$, $z \in \rho(H_{-,n})$, and $z \in \rho(H_{m,n}^w)$ for all $|m|$ sufficiently large by Lemma 8.16 and $\tilde{z} \neq z$. Hence, for all $|m|$ sufficiently large, the corresponding Weyl m-functions exist and $m_-^m(z,n) \to m_-(z,n)$ as $m \to -\infty$ by Lemma 2.32.
W.l.o.g. let $n = 0$ and let $c(z), s(z)$ denote a fundamental system of τ such that $c(z,-1) = 1, c(z,0) = 0$ and $s(z,-1) = 0, s(z,0) = 1$. Then, $u_-(z)$ is a linear combination of $c(z), s(z)$ and hence we have

$$u_-(j) = u_-(-1)c(j) + u_-(0)s(j) = u_-(0)(s(j) - a(-1)\frac{-u_-(-1)}{a(-1)u_-(0)}c(j))$$

$$= u_-(0)(s(j) - a(-1)m_-(z,0)c(j))$$

for all $j \in \mathbb{Z}$ by $m_-(z,0) = \langle \delta_{-1}, (H_{-,0} - z)^{-1}\delta_{-1}\rangle = -\frac{u_-(z,-1)}{a(-1)u_-(z,0)}$. Now, let $\phi_m(z)$ denote a solution of $(\tau_m - z)\phi_m(z) = 0$ such that $\phi_m(z,m) = 0$ and let $c_m(z), s_m(z)$ denote a fundamental system of τ_m such that $c_m(z,-1) = 1, c_m(z,0) = 0$ and $s_m(z,-1) = 0, s_m(z,0) = 1$. Then, $\phi_m(z)$ is a linear combination of $c_m(z), s_m(z)$ and hence we have

$$\begin{aligned}\phi_m(j) &= \phi_m(-1)c_m(j) + \phi_m(0)s_m(j)\\ &= \phi_m(0)(s_m(j) - a(-1)m_-^m(z,0)c_m(j))\end{aligned}$$

by $m_-^m(z,0) = \langle \delta_{-1}, (H^w_{m,0} - z)^{-1}\delta_{-1}\rangle = -\frac{\phi_m(z,-1)}{a(-1)\phi_m(z,0)}$. The difference equations τ_m and τ coincide at I for all $|m|$ sufficiently large and hence we have

$$\phi_m(j) = \phi_m(0)(s(j) - a(-1)m_-^m(z,0)c(j)) \quad \text{for all } j \in I.$$

By $\phi_m(0) \neq 0$ and $u_-(0) \neq 0$ there exists an $\alpha_m \neq 0$ such that $\varphi_m(z) = \alpha_m\phi_m(z)$ coincides with $u_-(z)$ at the point 0. Thus, for all $j \in I$ by

$$(\varphi_m - u_-)(j) = u_-(0)a(-1)c(j)(m_-(z,0) - m_-^m(z,0))$$

and $\lim_{\substack{m\to-\infty\\ m\in\mathscr{J}_w}} m_-^m(z,0) = m_-(z,0)$ we have $\lim_{\substack{m\to-\infty\\ m\in\mathscr{J}_w}} \varphi_m(j) = u_-(j)$. □

Now that we have seen that the solutions corresponding to the approximating semi-infinite problems converge to the Weyl solution (although we have a foreign boundary condition) on a *finite set* and it remains to ask if the number of nodes of the Wronskians coincide at some finite set.

And we will see that this number in general doesn't coincide and the problem arises from zeros of the Wronskian at the endpoints of the considered interval. Therefore one can e.g. think at a Wronskian which vanishes (and hence has -1 nodes at each finite set). Such a Wronskian can be approximated by a constant, nonvanishing Wronskian (which has 0 nodes on each finite interval).

To make this statement rigorously, in a first step we compare the number of nodes of the solutions itself, and we will see that we cannot loose nodes of solutions through approximation, since in the case of solutions we don't count zeros at the endpoints of the interval.

Lemma 8.24. *Let φ_n and φ be solutions of Jacobi difference equations such that $\varphi_n(j) \to \varphi(j)$ as $n \to \infty$ for all $j = k, \dots, l$, then we have*

$$\#_{(k,l)}(\varphi_n) \geqslant \#_{(k,l)}(\varphi)$$

for all n sufficiently large and moreover, $\#_{(k,l)}(\varphi_n) = \#_{(k,l)}(\varphi)$ if $\varphi(k), \varphi(l) \neq 0$.

Proof. Suppose $\varphi_n(m)$ and $\varphi(m)$ are of the same sign for all $m \in I$ where $\varphi(m) \neq 0$. If $\varphi(m)\varphi(m+1) \neq 0$, then either both solutions have a node at m or both solutions don't have a node at m. If $\varphi(m) = 0$, then by $\varphi(m-1)\varphi(m+1) < 0$ both solutions have exactly one node at $m-1$ and m. This proves the second claim. Now,

$$\#_{(k,l)}(\varphi_n) \geqslant \begin{cases} \#_{(k+1,l)}(\varphi_n) = \#_{(k+1,l)}(\varphi) & \text{if } \varphi(k) = 0, \varphi(l) \neq 0 \\ \#_{(k,l-1)}(\varphi_n) = \#_{(k,l-1)}(\varphi) & \text{if } \varphi(k) \neq 0, \varphi(l) = 0 \\ \#_{(k+1,l-1)}(\varphi_n) = \#_{(k+1,l-1)}(\varphi) & \text{if } \varphi(k) = 0, \varphi(l) = 0. \end{cases}$$

□

The key ingredient of the subsequent proof is, that also the Prüfer angles converge at a finite set, which is now shown.

Lemma 8.25. *Let φ_n and φ be solutions of Jacobi difference equations such that $\varphi_n(j) \to \varphi(j)$ as $n \to \infty$ for all $j = L-1, \dots, M+1$, then there exist corresponding Prüfer transformations such that*

$$\theta_{\varphi_n}(m) \to \theta_\varphi(m)$$

for all $m = L, \dots, M$.

Proof. Let n such that $\varphi_n(m)$ and $\varphi(m)$ are of the same sign at all $m \in I$ where $\varphi(m) \neq 0$ and let $\underline{m} = \min\{m \in I \mid \varphi(m) \neq 0\}$, i.e. $\underline{m} = L$ or $\underline{m} = L+1$. Consider the Prüfer transformations with base point $\underline{m}$, i,e. $\theta_\varphi(\underline{m}), \theta_{\varphi_n}(\underline{m}) \in (-\pi, \pi]$, then $\lfloor \theta_{\varphi_n}(\underline{m})/\pi \rfloor = \lfloor \theta_\varphi(\underline{m})/\pi \rfloor$ by $\varphi(\underline{m}) \neq 0$. Thus, $\theta_{\varphi_n}(\underline{m}) \to \theta_\varphi(\underline{m})$ by

$$\cot \theta_{\varphi_n}(\underline{m}) = \frac{-a(\underline{m})\varphi_n(\underline{m}+1)}{\varphi_n(\underline{m})} \to \cot \theta_\varphi(\underline{m}) = \frac{-a(\underline{m})\varphi(\underline{m}+1)}{\varphi(\underline{m})}.$$

Let $m = \underline{m}+1, \dots, M+1$ where $\varphi(m) \neq 0$, then

$$\begin{aligned} \lceil \theta_\varphi(m)/\pi \rceil &= \#_{(\underline{m},m)}(\varphi) + \lfloor \theta_\varphi(\underline{m})/\pi \rfloor + 1 \\ &= \#_{(\underline{m},m)}(\varphi_n) + \lfloor \theta_{\varphi_n}(\underline{m})/\pi \rfloor + 1 = \lceil \theta_{\varphi_n}(m)/\pi \rceil \end{aligned}$$

by Lemma 8.24 and

$$\cot \theta_{\varphi_n}(m) = \frac{-a(m)\varphi_n(m+1)}{\varphi_n(m)} \to \frac{-a(m)\varphi(m+1)}{\varphi(m)} = \cot \theta_\varphi(m),$$

thus $\theta_{\varphi_n}(m) \to \theta_\varphi(m)$. Now, let $m = \underline{m}+1, \dots, M$ such that $\varphi(m) = \rho_\varphi(m) \sin \theta_\varphi(m) = 0$, then there exists some $k \in \mathbb{Z}$ such that $\theta_\varphi(m) = k\pi$. Moreover, the solution φ_n has exactly one node at $m-1$ or m, hence by

$\theta_{\varphi_n}(m-1) \to k\pi - \frac{\pi}{2}$ we have $\theta_{\varphi_n}(m) \in (k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2})$. Now, $\theta_{\varphi_n}(m) \to \theta_{\varphi}(m)$ holds by

$$\tan\theta_{\varphi_n}(m) = \frac{\varphi_n(m)}{-a(m)\varphi_n(m+1)} \to \frac{\varphi(m)}{-a(m)\varphi(m+1)} = \tan\theta_{\varphi}(m).$$

□

In the last step we now establish the following inequalities for the number of nodes of the Wronskians and different counting methods. The result clearly depends on the behaviour of the Wronskian at the boundary. Note, that this also means that the difference cannot be large.

Lemma 8.26. *Let φ_n, φ be solutions of Jacobi difference equations such that $\varphi_n(j) \to \varphi(j)$ as $n \to \infty$ for all $j = L-1, \dots, M+1$, then for all n sufficiently large*

$$\#_{(L,M)}(\varphi_n, \phi) \geqslant \#_{(L,M]}(\varphi_n, \phi) \geqslant \#_{(L,M]}(\varphi, \phi), \tag{8.48}$$

$$\#_{[L,M]}(\varphi_n, \phi) \geqslant \#_{(L,M]}(\varphi_n, \phi) \geqslant \#_{(L,M]}(\varphi, \phi), \tag{8.49}$$

$$\#_{[L,M)}(\varphi_n, \phi) \leqslant \#_{[L,M)}(\varphi, \phi). \tag{8.50}$$

If $W_L(\varphi, \phi) \neq 0$ and $W_M(\varphi, \phi) = 0$, then

$$\#_{[L,M]}(\varphi_n, \phi) \geqslant \#_{[L,M]}(\varphi, \phi), \qquad \#_{(L,M)}(\varphi_n, \phi) \leqslant \#_{(L,M)}(\varphi, \phi). \tag{8.51}$$

If $W_L(\varphi, \phi) = 0$ and $W_M(\varphi, \phi) \neq 0$, then

$$\#_{[L,M]}(\varphi_n, \phi) \leqslant \#_{[L,M]}(\varphi, \phi), \qquad \#_{(L,M)}(\varphi_n, \phi) \geqslant \#_{(L,M)}(\varphi, \phi). \tag{8.52}$$

If $W_L(\varphi, \phi) \neq 0$ and $W_M(\varphi, \phi) \neq 0$ then we even have

$$\#_{[L,M]}(\varphi_n, \phi) = \#_{[L,M]}(\varphi, \phi), \qquad \#_{(L,M]}(\varphi_n, \phi) = \#_{(L,M]}(\varphi, \phi), \tag{8.53}$$

$$\#_{[L,M)}(\varphi_n, \phi) = \#_{[L,M)}(\varphi, \phi), \qquad \#_{(L,M)}(\varphi_n, \phi) = \#_{(L,M)}(\varphi, \phi). \tag{8.54}$$

The same holds for $W(\phi, \varphi)$.

Proof. Let n be sufficiently large and let $\theta_{\varphi}, \theta_{\varphi_n}$ be the Prüfer angles from Lemma 8.25. If $W_L(\varphi, \phi) \neq 0$, then by $(\theta_{\phi}(L) - \theta_{\varphi}(L))/\pi \notin \mathbb{Z}$ we have

$$\lceil(\theta_{\phi}(L) - \theta_{\varphi_n}(L))/\pi\rceil = \lceil(\theta_{\phi}(L) - \theta_{\varphi}(L))/\pi\rceil,$$
$$\lfloor(\theta_{\phi}(L) - \theta_{\varphi_n}(L))/\pi\rfloor = \lfloor(\theta_{\phi}(L) - \theta_{\varphi}(L))/\pi\rfloor.$$

The same holds at M. If $W_M(\varphi, \phi) = 0$, then

$$\lceil(\theta_{\phi}(M) - \theta_{\varphi_n}(M))/\pi\rceil \geqslant \lceil(\theta_{\phi}(M) - \theta_{\varphi}(M))/\pi\rceil,$$

$$\lfloor(\theta_\phi(M) - \theta_{\varphi_n}(M))/\pi\rfloor \leqslant \lfloor(\theta_\phi(M) - \theta_\varphi(M))/\pi\rfloor.$$

If $W_L(\varphi, \phi) = 0$, then

$$-\lceil(\theta_\phi(L) - \theta_{\varphi_n}(L))/\pi\rceil \leqslant -\lceil(\theta_\phi(L) - \theta_\varphi(L))/\pi\rceil = -\lfloor(\theta_\phi(L) - \theta_\varphi(L))/\pi\rfloor,$$
$$-\lfloor(\theta_\phi(L) - \theta_{\varphi_n}(L))/\pi\rfloor \geqslant -\lfloor(\theta_\phi(L) - \theta_\varphi(L))/\pi\rfloor,$$
$$-\lceil(\theta_\phi(L) - \theta_{\varphi_n}(L))/\pi\rceil \geqslant -\lfloor(\theta_\phi(L) - \theta_\varphi(L))/\pi\rfloor - 1.$$

Now use

$$\#_{[L,M]}(\varphi, \phi) = \lceil(\theta_\phi(M) - \theta_\varphi(M))/\pi\rceil - \lceil(\theta_\phi(L) - \theta_\varphi(L))/\pi\rceil,$$
$$\#_{(L,M)}(\varphi, \phi) = \lfloor(\theta_\phi(M) - \theta_\varphi(M))/\pi\rfloor - \lfloor(\theta_\phi(L) - \theta_\varphi(L))/\pi\rfloor,$$
$$\#_{(L,M]}(\varphi, \phi) = \lceil(\theta_\phi(M) - \theta_\varphi(M))/\pi\rceil - \lfloor(\theta_\phi(L) - \theta_\varphi(L))/\pi\rfloor - 1,$$
$$\#_{[L,M)}(\varphi, \phi) = \lfloor(\theta_\phi(M) - \theta_\varphi(M))/\pi\rfloor - \lceil(\theta_\phi(L) - \theta_\varphi(L))/\pi\rceil + 1.$$

□

Chapter 9

Below the essential spectra

In this chapter we establish the oscillation theorems for Wronskians below the essential spectrum of the corresponding operators, as already mentioned in the introduction. Therefore, as usual let u_- denote a solution fulfilling the left boundary condition of the corresponding operator. Hence, in the first part of this section, where semi-infinite operators H_+ are considered, u_- is a solution so that $u_-(0) = 0$ holds. And as soon as we look at H we assume $u_- \in \ell^2(-\mathbb{N})$.

Lemma 9.1. *Let $v = \tilde{u}_+(\lambda)$ be a Weyl solution of $(\tilde{\tau} - \lambda)u = 0$. Then, there exists an infinite subset $\mathscr{J}$ of $\mathscr{J}_v$ such that the family*

$$\{H_n^v\}_{n\in\mathscr{J}} \text{ is uniformly bounded.}$$

The same holds for $\{H_{m,+}^w\}_{m\in\mathscr{J}}$ where $w = \tilde{u}_-(\lambda)$.

Proof. Since v has only simple zeros $\mathscr{J}_v = \{n \in \mathbb{N}, n > 2 \mid v(n-1) \neq 0\}$ is an infinite set. If v has infinitely many zeros, then let

$$\mathscr{J} = \{n \in \mathbb{N}, n > 2 \mid v(n) = 0\}.$$

Thus, by $\frac{a(n-1)v(n)}{v(n-1)} = 0$ the family $\{H_n^v\}_{n\in\mathscr{J}}$ is uniformly bounded by $2\|a\|_\infty + \|b\|_\infty$. If v has only finitely many zeros, then fix some N so that $v(n) \neq 0$ for all $n \geqslant N$. By $\sum_{n=N}^\infty |v(n)|^2 < \infty$ and the ratio test $\liminf_{n\to\infty} \left|\frac{v(n)}{v(n-1)}\right| \leqslant 1$ holds. Now, let $\mathscr{J}$ be an infinite subset of $\mathscr{J}_v$ such that

$$\liminf_{n\to\infty} \left|\frac{v(n)}{v(n-1)}\right| = \lim_{\substack{n\to\infty\\ n\in\mathscr{J}}} \left|\frac{v(n)}{v(n-1)}\right| \leqslant 1.$$

Hence, $\lim_{\substack{n\to\infty\\ n\in\mathscr{J}}} \left|\frac{a(n-1)v(n)}{v(n-1)}\right| \leqslant \|a\|_\infty$ holds. Use reflection to obtain the second claim. □

9.1 The semi-infinite case

For all $z_\pm < \inf \sigma_{ess}(H_+^0)$ by Theorem 7.10 we have

$$\tau_0 - z_\pm \overset{rno_+}{\sim} \tau_1 - z_\pm \quad \text{and} \quad \tau_0 - z_\pm \overset{rno_+}{\sim} \tau_1 - z_\mp,$$

hence we already know that the Wronskians we consider in this section have at most finitely many nodes. Now, we state the precise connection between them and the spectra of the operators.

Theorem 9.2. *Let $z < \inf \sigma_{ess}(H_+^0)$. If $b_0 \downarrow b_1$ near ∞, then*

$$E_{(-\infty,z)}(H_+^1) - E_{(-\infty,z]}(H_+^0) = \#_{(0,\infty]}(u_{0,\pm}(z), u_{1,\mp}(z)) \tag{9.1}$$

holds, which is (1.18). If $b_0 \uparrow b_1$ near ∞, then

$$E_{(-\infty,z]}(H_+^1) - E_{(-\infty,z)}(H_+^0) = \#_{[0,\infty)}(u_{0,\pm}(z), u_{1,\mp}(z)). \tag{9.2}$$

Proof. For the first claim let $v = u_{0,+}(z)$ and by Lemma 9.1 there is some $\lambda < z$ less than the lower bound of $\mathcal{F} = \{H_+^0, H_+^1\} \cup \{H_n^{0,z}, H_n^{1,v}\}_{n\in\mathcal{J}}$. Then, by $z + (b_1 - b_0)(j) \in [\lambda, z]$ near ∞, Lemma 8.8, and Corollary 8.11 we have

$$\begin{aligned} &E_{(-\infty,z)}(H_+^1) - E_{(-\infty,z]}(H_+^0) = E_{(\lambda,z)}(H_+^1) - E_{(\lambda,z]}(H_+^0) \\ &= \lim_{n\to\infty}(E_{(\lambda,z)}(H_n^{1,v}) - E_{(\lambda,z]}(H_n^{0,z})) = \lim_{n\to\infty}(E_{(-\infty,z)}(H_n^{1,v}) - E_{(-\infty,z]}(H_n^{0,z})). \end{aligned}$$

Now use Lemma 8.19. For the second claim use $v = u_{1,+}(z)$, Corollary 8.9, $z + (b_1 - b_0)(m) \downarrow z$ near ∞, and Lemma 8.10.
For the third claim let $v = u_{0,+}(z)$ and again by Lemma 9.1 there is some $\lambda < z$ less than the lower bound of $\mathcal{F} = \{H_+^0, H_+^1\} \cup \{H_n^{0,z}, H_n^{1,v}\}_{n\in\mathcal{J}}$. By $z+(b_1-b_0)(m) \downarrow z$ near ∞, Lemma 8.10, Corollary 8.9, Theorem 1.5, $\tau_0 - z \overset{rno_+}{\sim} \tau_1 - z$, and Lemma 8.17 we obtain

$$\begin{aligned} E_{(\lambda,z]}(H_+^1) - E_{(\lambda,z)}(H_+^0) &= \lim_{n\to\infty}(E_{(\lambda,z]}(H_n^{1,v}) - E_{(\lambda,z)}(H_n^{0,z})) \\ &= \lim_{n\to\infty}(\#_{[0,n)}(\psi_{0,n,n}(z), \psi_{1,n,0}(z))) = \#_{[0,\infty)}(u_{0,+}(z), u_{1,-}(z)). \end{aligned}$$

Now, let $v = u_{1,+}(z)$ and consider

$$E_{(\lambda,z]}(H_n^{1,z}) - E_{(\lambda,z)}(H_n^{0,v}) = \#_{[0,n)}(\psi_{0,n,0}(z), \psi_{1,n,n}(z))$$

from Theorem 1.5. Then, we have $\lim_{n\to\infty} E_{(\lambda,z]}(H_n^{1,z}) = E_{(\lambda,z]}(H_+^1)$ and $\lim_{n\to\infty} E_{(\lambda,z)}(H_n^{0,v}) = E_{(\lambda,z)}(H_+^0)$ by Corollary 8.11 and Lemma 8.8, hence the last claim follows from Lemma 8.17. □

And moreover we find the following

Corollary 9.3. *Let* $z < \inf \sigma_{ess}(H^0_+)$.
If $b_0 \downarrow b_1$ *near* ∞, *then*

$$E_{(-\infty,z)}(H^1_+) - E_{(-\infty,z)}(H^0_+) = \#_{[0,\infty]}(u_{0,+}(z), u_{1,-}(z)), \tag{9.3}$$
$$E_{(-\infty,z]}(H^1_+) - E_{(-\infty,z]}(H^0_+) = \#_{[0,\infty]}(u_{0,-}(z), u_{1,+}(z)). \tag{9.4}$$

If $b_0 \uparrow b_1$ *near* ∞, *then*

$$E_{(-\infty,z)}(H^1_+) - E_{(-\infty,z)}(H^0_+) = \#_{(0,\infty)}(u_{0,-}(z), u_{1,+}(z)), \tag{9.5}$$
$$E_{(-\infty,z]}(H^1_+) - E_{(-\infty,z]}(H^0_+) = \#_{(0,\infty)}(u_{0,+}(z), u_{1,-}(z)). \tag{9.6}$$

Proof. Use Theorem 9.2 and

$$\begin{aligned}
\#_{[0,\infty]}(u_{0,+}(z), u_{1,-}(z)) &= \#_{(0,\infty]}(u_{0,+}(z), u_{1,-}(z)) + \begin{cases} 1 & \text{if } z \in \sigma(H^0_+) \\ 0 & \text{otherwise} \end{cases} \\
&= E_{(-\infty,z)}(H^1_+) - E_{(-\infty,z)}(H^0_+), \\
\#_{(0,\infty)}(u_{0,+}(z), u_{1,-}(z)) &= \#_{[0,\infty)}(u_{0,+}(z), u_{1,-}(z)) - \begin{cases} 1 & \text{if } z \in \sigma(H^0_+) \\ 0 & \text{otherwise} \end{cases} \\
&= E_{(-\infty,z]}(H^1_+) - E_{(-\infty,z]}(H^0_+)
\end{aligned}$$

to obtain the first and the last claim, the rest follows analogously. □

At last, we find a theorem for a Wronskian of solutions corresponding to two different spectral parameters.

Theorem 9.4. *Let* $z_- < z_+ < \inf \sigma_{ess}(H^0_+)$. *If* $b_0 \downarrow b_1$ *or* $b_0 \uparrow b_1$ *near* ∞, *then*

$$E_{(-\infty,z_+)}(H^1_+) - E_{(-\infty,z_-]}(H^0_+) = \#_{(0,\infty]}(u_{0,\pm}(z_-), u_{1,\mp}(z_+)), \tag{9.7}$$
$$E_{(-\infty,z_+)}(H^1_+) - E_{(-\infty,z_-)}(H^0_+) = \#_{[0,\infty]}(u_{0,+}(z_-), u_{1,-}(z_+)), \tag{9.8}$$
$$E_{(-\infty,z_+]}(H^1_+) - E_{(-\infty,z_-]}(H^0_+) = \#_{[0,\infty]}(u_{0,-}(z_-), u_{1,+}(z_+)), \tag{9.9}$$

and

$$E_{(-\infty,z_-]}(H^1_+) - E_{(-\infty,z_+)}(H^0_+) = \#_{[0,\infty]}(u_{0,\pm}(z_+), u_{1,\mp}(z_-)), \tag{9.10}$$
$$E_{(-\infty,z_-)}(H^1_+) - E_{(-\infty,z_+)}(H^0_+) = \#_{(0,\infty]}(u_{0,-}(z_+), u_{1,+}(z_-)), \tag{9.11}$$
$$E_{(-\infty,z_-]}(H^1_+) - E_{(-\infty,z_+]}(H^0_+) = \#_{(0,\infty]}(u_{0,+}(z_+), u_{1,-}(z_-)), \tag{9.12}$$

where $\#_{[0,\infty]}$ *can be replaced by* $\#_{[0,\infty)}$ *and* $\#_{(0,\infty]}$ *can be replaced by* $\#_{(0,\infty)}$ *and the Wronskians don't vanish near* $+\infty$ *by Lemma* 7.6.

Proof. We have $\tau_0 - z_\pm \overset{rno_+}{\sim} \tau_1 - z_\mp$ by Theorem 7.10. Let $v = u_{0,+}(z_-)$, $v = u_{1,+}(z_+)$, $v = u_{0,+}(z_+)$, or $v = u_{1,+}(z_-)$, then in either case by Lemma 9.1 there exists some infinite index set $\mathscr{J} \subseteq \mathscr{J}_v$ and some $\lambda < z_-$ less than the lower bound of $\mathcal{F} = \{H^0_+, H^1_+\} \cup \{H^{0,v}_n, H^{1,v}_n\}_{n\in\mathscr{J}}$. First, set $v = u_{0,+}(z_-)$ and consider

$$E_{(\lambda,z_+)}(H^{1,v}_n) - E_{(\lambda,z_-)}(H^{0,z_-}_n) = \#_{[0,n]}(\psi_{0,n,n}(z_-), \psi_{1,n,0}(z_+)),$$
$$E_{(\lambda,z_+)}(H^{1,v}_n) - E_{(\lambda,z_-]}(H^{0,z_-}_n) = \#_{(0,n]}(\psi_{0,n,n}(z_-), \psi_{1,n,0}(z_+))$$

from Theorem 1.5. By Lemma 8.8, Corollary 8.9, and Corollary 8.11 we have $\lim_{n\to\infty} E_{(\lambda,z_+)}(H^{1,v}_n) = E_{(\lambda,z_+)}(H^1_+)$, $\lim_{n\to\infty} E_{(\lambda,z_-)}(H^{0,z_-}_n) = E_{(\lambda,z_-)}(H^0_+)$ and $\lim_{n\to\infty} E_{(\lambda,z_-]}(H^{0,z_-}_n) = E_{(\lambda,z_-]}(H^0_+)$. Now, use Lemma 8.17 for the first and the third claim.
Next, set $v = u_{1,+}(z_+)$ and from Theorem 1.5 consider

$$E_{(\lambda,z_+]}(H^{1,z_+}_n) - E_{(\lambda,z_-]}(H^{0,v}_n) = \#_{[0,n]}(\psi_{0,n,0}(z_-), \psi_{1,n,n}(z_+)),$$
$$E_{(\lambda,z_+)}(H^{1,z_+}_n) - E_{(\lambda,z_-]}(H^{0,v}_n) = \#_{(0,n]}(\psi_{0,n,0}(z_-), \psi_{1,n,n}(z_+))$$

Then, by Corollary 8.11 and Corollary 8.9 we have $\lim_{n\to\infty} E_{(\lambda,z_+]}(H^{1,z_+}_n) = E_{(\lambda,z_+]}(H^1_+)$ and $\lim_{n\to\infty} E_{(\lambda,z_+)}(H^{1,z_+}_n) = E_{(\lambda,z_+)}(H^1_+)$. If $b_0 \downarrow b_1$, then $z_+ + b_0(m) - b_1(m) \downarrow z_+$, thus we have $\lim_{n\to\infty} E_{(\lambda,z_+]}(H^{0,v}_n) = E_{(\lambda,z_+]}(H^0_+)$ and $\lim_{n\to\infty} E_{(z_-,z_+]}(H^{0,v}_n) = E_{(z_-,z_+]}(H^0_+)$ by Lemma 8.10, hence by $E_{(\lambda,z_+]} - E_{(z_-,z_+]} = E_{(\lambda,z_-]}$ we have $\lim_{n\to\infty} E_{(\lambda,z_-]}(H^{0,v}_n) = E_{(\lambda,z_-]}(H^0_+)$. If $b_0 \uparrow b_1$, then $z_+ + b_0(m) - b_1(m) \uparrow z_+$, thus by Lemma 8.8 $\lim_{n\to\infty} E_{(\lambda,z_+)}(H^{0,v}_n) = E_{(\lambda,z_+)}(H^0_+)$ and $\lim_{n\to\infty} E_{(z_-,z_+)}(H^{0,v}_n) = E_{(z_-,z_+)}(H^0_+)$ holds and hence by $E_{(\lambda,z_+)} - E_{(z_-,z_+)} = E_{(\lambda,z_-]}$ we have $\lim_{n\to\infty} E_{(\lambda,z_-]}(H^{0,v}_n) = E_{(\lambda,z_-]}(H^0_+)$. Now, use Lemma 8.17 to obtain the second and the fourth claim.
Next, set $v = u_{0,+}(z_+)$ and consider

$$E_{(\lambda,z_-)}(H^{1,v}_n) - E_{(\lambda,z_+)}(H^{0,z_+}_n) = \#_{[0,n]}(\psi_{0,n,n}(z_+), \psi_{1,n,0}(z_-)),$$
$$E_{(\lambda,z_-)}(H^{1,v}_n) - E_{(\lambda,z_+]}(H^{0,z_+}_n) = \#_{(0,n]}(\psi_{0,n,n}(z_+), \psi_{1,n,0}(z_-))$$

from Theorem 1.5. We have $\lim_{n\to\infty} E_{(\lambda,z_+)}(H^{0,z_+}_n) = E_{(\lambda,z_+)}(H^0_+)$ and also $\lim_{n\to\infty} E_{(\lambda,z_+]}(H^{0,z_+}_n) = E_{(\lambda,z_+]}(H^0_+)$ by Corollary 8.9 and Corollary 8.11. If $b_0 \downarrow b_1$, then $z_+ + b_1(m) - b_0(m) \uparrow z_+$ near ∞, thus by Lemma 8.8 we have $\lim_{n\to\infty} E_{(\lambda,z_+)}(H^{1,v}_n) = E_{(\lambda,z_+)}(H^1_+)$, $\lim_{n\to\infty} E_{(z_-,z_+)}(H^{1,v}_n) = E_{(z_-,z_+)}(H^1_+)$. Thus, by $E_{(\lambda,z_+)} - E_{(z_-,z_+)} = E_{(\lambda,z_-]}$ and by Lemma 8.16 we have

$$\lim_{n\to\infty} E_{(\lambda,z_-)}(H^{1,v}_n) = \lim_{n\to\infty} E_{(\lambda,z_-]}(H^{1,v}_n) = E_{(\lambda,z_-]}(H^1_+).$$

If $b_0 \uparrow b_1$, then $z_+ + b_1(m) - b_0(m) \downarrow z_+$ near ∞, thus by Lemma 8.10 we have $\lim_{n\to\infty} E_{(\lambda,z_+]}(H^{1,v}_n) = E_{(\lambda,z_+]}(H^1_+)$ and $\lim_{n\to\infty} E_{(z_-,z_+]}(H^{1,v}_n) = E_{(z_-,z_+]}(H^1_+)$. Thus, by $E_{(\lambda,z_+]} - E_{(z_-,z_+]} = E_{(\lambda,z_-]}$ and Lemma 8.16 we have

$$\lim_{n\to\infty} E_{(\lambda,z_-)}(H^{1,v}_n) = \lim_{n\to\infty} E_{(\lambda,z_-]}(H^{1,v}_n) = E_{(\lambda,z_-]}(H^1_+).$$

Hence, Lemma 8.17 proves the fifth and the eighth claim.
Set $v = u_{1,+}(z_-)$ and consider

$$E_{(\lambda,z_-]}(H_n^{1,z_-}) - E_{(\lambda,z_+]}(H_n^{0,v}) = \#_{[0,n]}(\psi_{0,n,0}(z_+), \psi_{1,n,n}(z_-)),$$
$$E_{(\lambda,z_-)}(H_n^{1,z_-}) - E_{(\lambda,z_+]}(H_n^{0,v}) = \#_{(0,n]}(\psi_{0,n,0}(z_+), \psi_{1,n,n}(z_-))$$

from Theorem 1.5. We have $\lim_{n\to\infty} E_{(\lambda,z_-)}(H_n^{1,z_-}) = E_{(\lambda,z_-)}(H_+^1)$ and also $\lim_{n\to\infty} E_{(\lambda,z_-]}(H_n^{1,z_-}) = E_{(\lambda,z_-]}(H_+^1)$ by Corollary 8.9 and Corollary 8.11. By Lemma 8.8 and Lemma 8.16 we have

$$\lim_{n\to\infty} E_{(\lambda,z_+]}(H_n^{0,v}) = \lim_{n\to\infty} E_{(\lambda,z_+)}(H_n^{0,v}) = E_{(\lambda,z_+)}(H_+^0).$$

Now, use Lemma 8.17 again. □

9.2 The infinite case

As already discussed earlier we have

$$\tau_0 - z_\pm \overset{rno}{\sim} \tau_1 - z_\pm \quad \text{and} \quad \tau_0 - z_\pm \overset{rno}{\sim} \tau_1 - z_\mp$$

if $z_\pm < \inf \sigma_{ess}(H_0)$, see Theorem 7.11. Thus, below the essential spectrum of infinite Jacobi operators we obtain the following

Theorem 9.5. *Let $z < \inf \sigma_{ess}(H_0)$. If $b_0 \downarrow b_1$ near $+\infty$ and near $-\infty$, then*

$$E_{(-\infty,z)}(H_1) - E_{(-\infty,z]}(H_0) = \#_{(-\infty,\infty]}(u_{0,\pm}(z), u_{1,\mp}(z)) \tag{9.13}$$

which is (1.14). If $b_0 \downarrow b_1$ near $+\infty$ and $b_0 \uparrow b_1$ near $-\infty$, then

$$E_{(-\infty,z)}(H_1) - E_{(-\infty,z)}(H_0) = \#_{[-\infty,\infty]}(u_{0,+}(z), u_{1,-}(z)), \tag{9.14}$$
$$E_{(-\infty,z]}(H_1) - E_{(-\infty,z]}(H_0) = \#_{[-\infty,\infty]}(u_{0,-}(z), u_{1,+}(z)). \tag{9.15}$$

If $b_0 \uparrow b_1$ near $+\infty$ and $b_0 \downarrow b_1$ near $-\infty$, then

$$E_{(-\infty,z)}(H_1) - E_{(-\infty,z)}(H_0) = \#_{(-\infty,\infty)}(u_{0,-}(z), u_{1,+}(z)), \tag{9.16}$$
$$E_{(-\infty,z]}(H_1) - E_{(-\infty,z]}(H_0) = \#_{(-\infty,\infty)}(u_{0,+}(z), u_{1,-}(z)). \tag{9.17}$$

If $b_0 \uparrow b_1$ near $+\infty$ and near $-\infty$, then

$$E_{(-\infty,z]}(H_1) - E_{(-\infty,z)}(H_0) = \#_{[-\infty,\infty)}(u_{0,\pm}(z), u_{1,\mp}(z)). \tag{9.18}$$

Proof. Let $w = u_{0,-}(z)$ or $w = u_{1,-}(z)$, then by Lemma 9.1 there is some infinite set $\mathscr{J} \subseteq \mathscr{J}_w$ and some $\lambda < z$ less than the lower bound of $\mathcal{F} = \{H_0, H_1\} \cup \{H_{m,+}^{0,w}, H_{m,+}^{1,w}\}_{n\in\mathscr{J}}$. We assume $n \in \mathscr{J}$ and let $w = u_{0,-}(z)$ at first.

If $b_0 \downarrow b_1$ near $-\infty$, then by Lemma 8.8 $\lim_{m\to-\infty} E_{(\lambda,z)}(H^{1,w}_{m,+}) = E_{(\lambda,z)}(H_1)$ holds. If $b_0 \uparrow b_1$ near $-\infty$, then $\lim_{m\to-\infty} E_{(\lambda,z]}(H^{1,w}_{m,+}) = E_{(\lambda,z]}(H_1)$ holds by Lemma 8.10. If $b_0 \downarrow b_1$ near ∞, then by Corollary 8.11, Theorem 9.2, and Lemma 8.20 we have

$$\begin{aligned} E_{(-\infty,z)}(H_1) - E_{(-\infty,z]}(H_0) &= \lim_{m\to-\infty}(E_{(-\infty,z)}(H^{1,z}_{m,+}) - E_{(-\infty,z]}(H^{0,z}_{m,+})) \\ &= \lim_{m\to-\infty} \#_{(m,\infty]}(\psi_{0,m,m}(z), \psi_{1,m,+}(z)) = \#_{(-\infty,\infty]}(u_{0,-}(z), u_{1,+}(z)) \end{aligned}$$

if $b_0 \downarrow b_1$ near $-\infty$ and moreover by Corollary 9.3 we have

$$\begin{aligned} E_{(-\infty,z]}(H_1) - E_{(-\infty,z]}(H_0) &= \lim_{m\to-\infty}(E_{(-\infty,z]}(H^{1,z}_{m,+}) - E_{(-\infty,z]}(H^{0,z}_{m,+})) \\ &= \lim_{m\to-\infty} \#_{[m,\infty]}(\psi_{0,m,m}(z), \psi_{1,m,+}(z)) = \#_{[-\infty,\infty]}(u_{0,-}(z), u_{1,+}(z)) \end{aligned}$$

if $b_0 \uparrow b_1$ near $-\infty$.
If $b_0 \uparrow b_1$ near ∞, then by Corollary 8.9, Corollary 9.3, and Lemma 8.20 we have

$$\begin{aligned} E_{(-\infty,z)}(H_1) - E_{(-\infty,z)}(H_0) &= \lim_{m\to-\infty}(E_{(-\infty,z)}(H^{1,z}_{m,+}) - E_{(-\infty,z)}(H^{0,z}_{m,+})) \\ &= \lim_{m\to-\infty} \#_{(m,\infty)}(\psi_{0,m,m}(z), \psi_{1,m,+}(z)) = \#_{(-\infty,\infty)}(u_{0,-}(z), u_{1,+}(z)) \end{aligned}$$

if $b_0 \downarrow b_1$ near $-\infty$ and moreover by Corollary 9.3 we have

$$\begin{aligned} E_{(-\infty,z]}(H_1) - E_{(-\infty,z)}(H_0) &= \lim_{m\to-\infty}(E_{(-\infty,z]}(H^{1,z}_{m,+}) - E_{(-\infty,z)}(H^{0,z}_{m,+})) \\ &= \lim_{m\to-\infty} \#_{[m,\infty)}(\psi_{0,m,m}(z), \psi_{1,m,+}(z)) = \#_{[-\infty,\infty)}(u_{0,-}(z), u_{1,+}(z)) \end{aligned}$$

if $b_0 \uparrow b_1$ near $-\infty$. This proves the first part.
For the rest now set $w = u_{1,-}(z)$. Then, by Corollary 8.9 and Corollary 8.11 we have $\lim_{m\to-\infty} E_{(\lambda,z)}(H^{1,z}_{m,+}) = E_{(\lambda,z)}(H_1)$ and $\lim_{m\to-\infty} E_{(\lambda,z]}(H^{1,z}_{m,+}) = E_{(\lambda,z]}(H_1)$. By Theorem 7.11 we have $\tau_0 - z \overset{rno}{\sim} \tau_1 - z$ and hence by Lemma 8.20

$$\lim_{m\to-\infty} \#_{[m,\infty]}(\psi_{0,m,+}(z), \psi_{1,m,m}(z)) = \#_{[-\infty,\infty]}(u_{0,+}(z), u_{1,-}(z))$$

holds, where $\#_{[0,\cdot]}$ can be replaced by $\#_{(0,\cdot]}$, $\#_{[0,\cdot)}$, or $\#_{(0,\cdot)}$. From Theorem 9.2 we obtain

$$\begin{aligned} E_{(-\infty,z)}(H^{1,z}_{m,+}) - E_{(-\infty,z)}(H^{0,w}_{m,+}) &= \#_{[m,\infty]}(\psi_{0,m,+}(z), \psi_{1,m,m}(z)), \\ E_{(-\infty,z)}(H^{1,z}_{m,+}) - E_{(-\infty,z]}(H^{0,w}_{m,+}) &= \#_{(m,\infty]}(\psi_{0,m,+}(z), \psi_{1,m,m}(z)) \end{aligned}$$

if $b_0 \downarrow b_1$ near ∞ and

$$E_{(-\infty,z]}(H^{1,z}_{m,+}) - E_{(-\infty,z)}(H^{0,w}_{m,+}) = \#_{[m,\infty)}(\psi_{0,m,+}(z), \psi_{1,m,m}(z)),$$

$$E_{(-\infty,z]}(H^{1,z}_{m,+}) - E_{(-\infty,z]}(H^{0,w}_{m,+}) = \#_{(m,\infty)}(\psi_{0,m,+}(z), \psi_{1,m,m}(z))$$

if $b_0 \uparrow b_1$ near ∞. If $b_0 \downarrow b_1$ near $-\infty$, then $z + b_0(m) - b_1(m) \downarrow z$ near $-\infty$, thus by Lemma 8.10 $\lim_{m\to-\infty} E_{(\lambda,z]}(H^{0,w}_{m,+}) = E_{(\lambda,z]}(H_0)$ holds and if $b_0 \uparrow b_1$ near $-\infty$, then $z + b_0(m) - b_1(m) \uparrow z$ near $-\infty$, hence by Lemma 8.8 we have $\lim_{m\to-\infty} E_{(\lambda,z)}(H^{0,w}_{m,+}) = E_{(\lambda,z)}(H_0)$. □

In the last step we now investigate the Wronskian of solutions at z_- and z_+ on the line.

Theorem 9.6. *Let $z_- < z_+ < \inf \sigma_{ess}(H_0)$. If $b_0 \downarrow b_1$ or $b_0 \uparrow b_1$ near $+\infty$ and $b_0 \downarrow b_1$ or $b_0 \uparrow b_1$ near $-\infty$, then*

$$E_{(-\infty,z_+)}(H_1) - E_{(-\infty,z_-]}(H_0) = \#_{[-\infty,\infty]}(u_{0,\pm}(z_-), u_{1,\mp}(z_+)), \tag{9.19}$$
$$E_{(-\infty,z_-]}(H_1) - E_{(-\infty,z_+)}(H_0) = \#_{[-\infty,\infty]}(u_{0,\pm}(z_+), u_{1,\mp}(z_-)), \tag{9.20}$$

where the Wronskians don't vanish near $\pm\infty$ by Lemma 7.6, thus $\#_{[-\infty,\infty]}$ can be replaced by $\#_{(-\infty,\infty]}$, $\#_{[-\infty,\infty)}$, or $\#_{(-\infty,\infty)}$.

Proof. We have $\tau_0 - z_\pm \overset{rno}{\sim} \tau_1 - z_\mp$ by Theorem 7.11. If $w = u_{0,-}(z_\pm)$ or $w = u_{1,-}(z_\pm)$, then by Lemma 9.1 there is some infinite set $\mathscr{J} \subseteq \mathscr{J}_w$ and some $\lambda < z_-$ less than the lower bound of $\mathcal{F} = \{H_0, H_1\} \cup \{H^{0,w}_{m,+}, H^{1,w}_{m,+}\}_{m\in\mathscr{J}}$.
For the first claim set $w = u_{1,-}(z_+)$: if $b_0 \downarrow b_1$ near $-\infty$, then $z_+ + b_0(m) - b_1(m) \downarrow z_+$, thus by Lemma 8.10 $\lim_{m\to-\infty} E_{(\lambda,z_+]}(H^{0,w}_{m,+}) = E_{(\lambda,z_+]}(H_0)$ and $\lim_{m\to-\infty} E_{(z_-,z_+]}(H^{0,w}_{m,+}) = E_{(z_-,z_+]}(H_0)$ holds. By $E_{(\lambda,z_+]} - E_{(z_-,z_+]} = E_{(\lambda,z_-]}$ and Lemma 8.16 we have

$$\lim_{m\to-\infty} E_{(\lambda,z_-)}(H^{0,w}_{m,+}) = \lim_{m\to-\infty} E_{(\lambda,z_-]}(H^{0,w}_{m,+}) = E_{(\lambda,z_-]}(H_0).$$

If $b_0 \uparrow b_1$, then $z_+ + b_0(m) - b_1(m) \uparrow z_+$, hence by Lemma 8.8 we have $\lim_{m\to-\infty} E_{(\lambda,z_+)}(H^{0,w}_{m,+}) = E_{(\lambda,z_+)}(H_0)$ as well as $\lim_{m\to-\infty} E_{(z_-,z_+)}(H^{0,w}_{m,+}) = E_{(z_-,z_+)}(H_0)$ and by $E_{(\lambda,z_+)} - E_{(z_-,z_+)} = E_{(\lambda,z_-]}$ and Lemma 8.16 we again have $\lim_{m\to-\infty} E_{(\lambda,z_-)}(H^{0,w}_{m,+}) = \lim_{m\to-\infty} E_{(\lambda,z_-]}(H^{0,w}_{m,+}) = E_{(\lambda,z_-]}(H_0)$. Now, Corollary 8.9 implies $\lim_{m\to-\infty} E_{(\lambda,z_+)}(H^{1,z_+}_{m,+}) = E_{(\lambda,z_+)}(H_1)$ and from Theorem 9.4 in any case (if $b_0 \downarrow b_1$ or $b_0 \uparrow b_1$ near $+\infty$) we obtain

$$E_{(-\infty,z_+)}(H^{1,z_+}_{m,+}) - E_{(-\infty,z_-)}(H^{0,w}_{m,+}) = \#_{[m,\infty]}(\psi_{0,m,+}(z_-).\psi_{1,m,m}(z_+))$$

Now, use Lemma 8.20.
For the second claim set $w = u_{0,-}(z_-)$, then we have $\lim_{m\to-\infty} E_{(\lambda,z_-]}(H^{0,z_-}_{m,+}) = E_{(\lambda,z_-]}(H_0)$ by Corollary 8.11, and by Lemma 8.16 and Lemma 8.8 we have

$$\lim_{m\to-\infty} E_{(\lambda,z_+]}(H^{1,w}_{m,+}) = \lim_{m\to-\infty} E_{(\lambda,z_+)}(H^{1,w}_{m,+}) = E_{(\lambda,z_+)}(H_1).$$

From Theorem 9.4 we obtain

$$E_{(-\infty,z_+]}(H^{1,w}_{m,+}) - E_{(-\infty,z_-]}(H^{0,z_-}_{m,+}) = \#_{[m,\infty]}(\psi_{0,m,m}(z_-), \psi_{1,m,+}(z_+))$$

if $b_0 \downarrow b_1$ or $b_0 \uparrow b_1$ near ∞, hence the claim follows from Lemma 8.20.
Now, set $w = u_{1,-}(z_-)$, then by Corollary 8.11, Lemma 8.8, Theorem 9.4, and Lemma 8.20 we have

$$E_{(\lambda,z_-]}(H_1) - E_{(\lambda,z_+)}(H_0) = \lim_{m\to-\infty}(E_{(-\infty,z_-]}(H^{1,z_-}_{m,+}) - E_{(-\infty,z_+)}(H^{0,w}_{m,+}))$$
$$= \lim_{m\to-\infty} \#_{[m,\infty]}(\psi_{0,m,+}(z_+), \psi_{1,m,m}(z_-)) = \#_{[-\infty,\infty]}(u_{0,+}(z_+), u_{1,-}(z_-))$$

if $b_0 \downarrow b_1$ or $b_0 \uparrow b_1$ near $+\infty$.
For the last claim set $w = u_{0,-}(z_+)$. If $b_0 \downarrow b_1$ near $-\infty$, then $z_+ + b_1(m) - b_0(m) \uparrow z_+$ near $-\infty$, thus by Lemma 8.8 we have $\lim_{m\to-\infty} E_{(\lambda,z_+)}(H^{1,w}_{m,+}) = E_{(\lambda,z_+)}(H_1)$, $\lim_{m\to-\infty} E_{(z_-,z_+)}(H^{1,w}_{m,+}) = E_{(z_-,z_+)}(H_1)$. Hence, by $E_{(\lambda,z_+)} - E_{(z_-,z_+)} = E_{(\lambda,z_-]}$ we have $\lim_{m\to-\infty} E_{(\lambda,z_-]}(H^{1,w}_{m,+}) = E_{(\lambda,z_-]}(H_1)$. If $b_0 \uparrow b_1$ near $-\infty$, then $z_+ + b_1(m) - b_0(m) \downarrow z_+$ near $-\infty$, thus by Lemma 8.10 we have $\lim_{m\to-\infty} E_{(\lambda,z_+]}(H^{1,w}_{m,+}) = E_{(\lambda,z_+]}(H_1)$, $\lim_{m\to-\infty} E_{(z_-,z_+]}(H^{1,w}_{m,+}) = E_{(z_-,z_+]}(H_1)$. Hence, $\lim_{m\to-\infty} E_{(\lambda,z_-]}(H^{1,w}_{m,+}) = E_{(\lambda,z_-]}(H_1)$ holds by $E_{(\lambda,z_+]} - E_{(z_-,z_+]} = E_{(\lambda,z_-]}$. By Theorem 9.4 we have

$$E_{(-\infty,z_-]}(H^{1,w}_{m,+}) - E_{(-\infty,z_+)}(H^{0,z_+}_{m,+}) = \#_{[m,\infty]}(\psi_{0,m,m}(z_+), \psi_{1,m,+}(z_-))$$

if $b_0 \downarrow b_1$ or $b_0 \uparrow b_1$ near $+\infty$. Hence, we have $\lim_{m\to-\infty} E_{(\lambda,z_+)}(H^{0,z_+}_{m,+}) = E_{(\lambda,z_+)}(H_0)$ by Corollary 8.9, and Lemma 8.20 proves the claim. $\square$

Chapter 10

Semi-infinite Jacobi operators

In this chapter we consider gaps of the essential spectrum of semi-infinite Jacobi operators to prove Theorem 1.2.
First of all we briefly recall the renormalized oscillation theorem from [46], where one single Jacobi operator is considered. In contrast thereto, we investigate Wronskians which consist of solutions of two different Jacobi operators.

Theorem 10.1 (Renormalized oscillation theorem). *[42, Theorem 4.17]*
Let $[z_-, z_+] \cap \sigma_{ess}(H_+) = \emptyset$*, then*

$$E_{(z_-,z_+)}(H_+) = \#_{(0,\infty]}(u_-(z_-), u_-(z_+)).$$

If we look at only one operator, then we easily also obtain the following theorem from our previous considerations.

Theorem 10.2. *Let* $[z_-, z_+] \cap \sigma_{ess}(H_+) = \emptyset$*, then*

$$\begin{aligned} E_{(z_-,z_+)}(H_+) &= \#_{(0,\infty]}(u_\pm(z_-), u_\mp(z_+)), \\ E_{[z_-,z_+)}(H_+) &= \#_{[0,\infty]}(u_+(z_-), u_-(z_+)), \\ E_{(z_-,z_+]}(H_+) &= \#_{[0,\infty]}(u_-(z_-), u_+(z_+)), \end{aligned}$$

where the Wronskians don't vanish near ∞*, that is,* $\#_{(0,\infty]}$ *can be replaced by* $\#_{(0,\infty)}$ *and* $\#_{[0,\infty]}$ *by* $\#_{[0,\infty)}$*.*

Proof. By Lemma 7.6 the Wronskian cannot vanish near ∞. First, let $v = u_+(z_-)$, where $n \in \mathscr{J}_v$, then by Theorem 1.5, Lemma 8.8, Lemma 8.8, and Lemma 8.17

$$E_{[z_-,z_+)}(H_+) = \lim_{n\to\infty} E_{[z_-,z_+)}(H_n^v)$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \#_{[0,n]}(\psi_{n,n}(z_-), \psi_{n,0}(z_+)) = \#_{[0,\infty]}(u_+(z_-), u_-(z_+)),\\
E_{(z_-,z_+)}(H_+) &= \lim_{n\to\infty} E_{(z_-,z_+)}(H_n^v)\\
&= \lim_{n\to\infty} \#_{(0,n]}(\psi_{n,n}(z_-), \psi_{n,0}(z_+)) = \#_{(0,\infty]}(u_+(z_-), u_-(z_+))
\end{aligned}
$$

holds. Now, let $v = u_+(z_+)$, then we find analogously

$$
\begin{aligned}
E_{(z_-,z_+]}(H_+) &= \lim_{n\to\infty} E_{(z_-,z_+]}(H_n^v)\\
&= \lim_{n\to\infty} \#_{[0,n]}(\psi_{n,0}(z_-), \psi_{n,n}(z_+)) = \#_{[0,\infty]}(u_-(z_-), u_+(z_+)),\\
E_{(z_-,z_+)}(H_+) &= \lim_{n\to\infty} E_{(z_-,z_+)}(H_n^v)\\
&= \lim_{n\to\infty} \#_{(0,n]}(\psi_{n,0}(z_-), \psi_{n,n}(z_+)) = \#_{(0,\infty]}(u_-(z_-), u_+(z_+)).
\end{aligned}
$$

□

10.1 A first theorem on the half-line

Now we turn toward the investigation of to different Jacobi operators H_+^0 and H_+^1. From now on we assume

$$
[z_-, z_+] \cap \sigma_{ess}(H_+) = \emptyset, \quad z_- < z_+, \tag{10.1}
$$

$$
a = a_0 = a_1, \quad \text{and} \quad b_0 \downarrow b_1 \text{ near } \infty. \tag{10.2}
$$

We remark, that the notation used in this section has been introduced in Subsection 8.2.1. Additionally we abbreviate

$$
N_0(z) = \#_{(0,\infty]}(u_{0,+}(z), u_{1,-}(z)), \quad N_1(z) = \#_{(0,\infty]}(u_{0,-}(z), u_{1,+}(z)). \tag{10.3}
$$

Both numbers are finite for all $z \notin \sigma_{ess}(H_+^0)$ by Theorem 7.10.

Lemma 10.3. *Let $b_0 \downarrow b_1$ near ∞, $z \in [\lambda_0, \lambda_1] \cap \sigma_{ess}(H_+^0) = \emptyset$, and let $v = u_{j,+}(z)$, $j = 0, 1$. Then, for all $\lambda \in [\lambda_0, \lambda_1]$ there exists an N_λ and a constant $C_{j,z}(\lambda)$ so that*

$$
E_{(-\infty,\lambda)}(H_n^{1,v}) - E_{(-\infty,\lambda]}(H_n^{0,v}) = C_{j,z}(\lambda) \geqslant N_j(\lambda)
$$

holds for all $n \geqslant N_\lambda$, $n \in \mathscr{J}_v$. Moreover,

$$
C_{0,z}(\lambda) - N_0(z) = C_{1,z}(\lambda) - N_1(z) = \begin{cases} E_{[z,\lambda)}(H_+^1) - E_{(z,\lambda)}(H_+^0) & \text{if } \lambda > z\\ 0 & \text{if } \lambda = z\\ -E_{(\lambda,z)}(H_+^1) + E_{(\lambda,z]}(H_+^0) & \text{if } \lambda < z \end{cases}
$$

and $N_1(\lambda) \leqslant C_{0,z}(\lambda)$, $N_0(\lambda) \leqslant C_{1,z}(\lambda)$ holds if $\lambda \neq z$.

Proof. Let $v = u_{0,+}(z)$ and $n \in \mathscr{J}_v$ be sufficiently large. If $\lambda = z$, then the first claim holds by Lemma 8.19. If $\lambda < z$, then by Corollary 8.11, $\lambda \notin \sigma(H_n^{1,v})$ (Lemma 8.16), $z + b_1 - b_0 \uparrow z$, and Lemma 8.8

$$\begin{aligned} E_{(\lambda,z]}(H_n^{0,z}) &= E_{(\lambda,z]}(H_+^0) = M_0 < \infty, \\ E_{[\lambda,z)}(H_n^{1,v}) &= E_{(\lambda,z)}(H_n^{1,v}) = E_{(\lambda,z)}(H_+^1) = M_1 < \infty \end{aligned}$$

holds. Now,

$$\begin{aligned} M_1 - M_0 &= E_{[\lambda,z)}(H_n^{1,v}) - E_{(\lambda,z]}(H_n^{0,z}) \\ &= \underbrace{E_{(-\infty,z)}(H_n^{1,v}) - E_{(-\infty,z]}(H_n^{0,z})}_{N_0(z)} - (\underbrace{E_{(-\infty,\lambda)}(H_n^{1,v}) - E_{(-\infty,\lambda]}(H_n^{0,z})}_{C_{0,z}(\lambda)}), \end{aligned}$$

hence $C_{0,z}(\lambda) - N_0(z) = -E_{(\lambda,z)}(H_+^1) + E_{(\lambda,z]}(H_+^0)$.
If $\lambda > z$, then by Lemma 8.16, Corollary 8.9, $z + b_1 - b_0 \uparrow z$, and Lemma 8.10

$$\begin{aligned} E_{(z,\lambda]}(H_n^{0,z}) &= E_{(z,\lambda)}(H_n^{0,z}) = E_{(z,\lambda)}(H_+^0) = \tilde{M}_0, \\ E_{[z,\lambda)}(H_n^{1,v}) &= E_{[z,\lambda)}(H_+^1) = \tilde{M}_1, \end{aligned}$$

holds and thus,

$$\begin{aligned} \tilde{M}_1 - \tilde{M}_0 &= E_{[z,\lambda)}(H_n^{1,v}) - E_{(z,\lambda]}(H_n^{0,z}) \\ &= \underbrace{E_{(-\infty,\lambda)}(H_n^{1,v}) - E_{(-\infty,\lambda]}(H_n^{0,z})}_{C_{0,z}(\lambda)} - (\underbrace{E_{(-\infty,z)}(H_n^{1,v}) - E_{(-\infty,z]}(H_n^{0,z})}_{N_0(z)}) \end{aligned}$$

hence $C_{0,z}(\lambda) - N_0(z) = E_{[z,\lambda)}(H_+^1) - E_{(z,\lambda)}(H_+^0)$. If $\lambda \neq z$, then let K such that $(b_0 - b_1)(j) \geqslant 0$ for all $j > K$ and all nodes of $W(u_{0,+}(\lambda), u_{1,-}(\lambda))$ and $W(u_{0,-}(\lambda), u_{1,+}(\lambda))$ are to the left of K. Let $j = 0, 1$, then by Lemma 8.21 there exist solutions $\varphi_{j,n}(\lambda)$ of $(\tau_{j,n} - \lambda)u = 0$ such that

$$\varphi_{j,n}(\lambda, n) = 0 \quad \text{and} \quad \varphi_{j,n}(\lambda, m) \to u_{j,+}(\lambda, m) \quad \text{at } m = -1, \dots, K+2.$$

Let $\psi_{j,n,0}(\lambda)$ denote a solution of $(\tau_{j,n} - \lambda)u = 0$ vanishing at the point 0. The solution $u_{j,-}(\lambda)$ also is a solution of $(\tau_{j,n} - \lambda)u = 0$ below n and hence by Lemma 8.26 for all n sufficiently large we have

$$\begin{aligned} C_{0,z}(\lambda) &= \#_{(0,n]}(\varphi_{0,n}(\lambda), \psi_{1,n,0}(\lambda)) \geqslant \#_{(0,K+1]}(\varphi_{0,n}(\lambda), \psi_{1,n,0}(\lambda)) \\ &= \#_{(0,K+1]}(\varphi_{0,n}(\lambda), u_{1,-}(\lambda)) \geqslant N_0(\lambda), \\ C_{0,z}(\lambda) &= \#_{(0,n]}(\psi_{0,n,0}(\lambda), \varphi_{1,n}(\lambda)) \geqslant \#_{(0,K+1]}(u_{0,-}(\lambda), \varphi_{1,n}(\lambda)) \geqslant N_1(\lambda). \end{aligned}$$

Now, let $v = u_{1,+}(z)$ and $n \in \mathscr{J}_v$. Then, if $\lambda = z$, the claim holds by

Lemma 8.19. Let $\lambda < z$, then

$$E_{(\lambda,z]}(H_n^{0,v}) = E_{(\lambda,z]}(H_+^0) = M_0,$$
$$E_{[\lambda,z)}(H_n^{1,z}) = E_{(\lambda,z)}(H_n^{1,z}) = E_{(\lambda,z)}(H_+^1) = M_1$$

holds for all n sufficiently large, where we used $z + b_0 - b_1 \downarrow z$, Lemma 8.10, Lemma 8.16, and Lemma 8.8. Now,

$$\begin{aligned} M_1 - M_0 &= E_{[\lambda,z)}(H_n^{1,z}) - E_{(\lambda,z]}(H_n^{0,v}) \\ &= \underbrace{E_{(-\infty,z)}(H_n^{1,z}) - E_{(-\infty,z]}(H_n^{0,v})}_{N_1(z)} - (\underbrace{E_{(-\infty,\lambda)}(H_n^{1,z}) - E_{(-\infty,\lambda]}(H_n^{0,v})}_{C_{1,z}(\lambda)}), \end{aligned}$$

hence $C_{1,z}(\lambda) - N_1(z) = -E_{(\lambda,z)}(H_+^1) + E_{(\lambda,z]}(H_+^0)$. And if $\lambda > z$, then

$$E_{(z,\lambda]}(H_n^{0,v}) = E_{(z,\lambda)}(H_n^{0,v}) = E_{(z,\lambda)}(H_+^0) = \tilde{M}_0,$$
$$E_{[z,\lambda)}(H_n^{1,z}) = E_{[z,\lambda)}(H_+^1) = \tilde{M}_1$$

holds, where we used Lemma 8.16, $z+b_0-b_1 \downarrow z$, Lemma 8.8, and Corollary 8.11. Thus,

$$\begin{aligned} \tilde{M}_1 - \tilde{M}_0 &= E_{[z,\lambda)}(H_n^{1,z}) - E_{(z,\lambda]}(H_n^{0,v}) \\ &= \underbrace{E_{(-\infty,\lambda)}(H_n^{1,z}) - E_{(-\infty,\lambda]}(H_n^{0,v})}_{C_{1,z}(\lambda)} - (\underbrace{E_{(-\infty,z)}(H_n^{1,z}) - E_{(-\infty,z]}(H_n^{0,v})}_{N_1(z)}) \end{aligned}$$

implies $C_{1,z}(\lambda) = N_1(z) + E_{[z,\lambda)}(H_+^1) - E_{(z,\lambda)}(H_+^0)$. With exactly the same argument as in the previous case for all $\lambda \neq z$ we obtain

$$C_{1,z}(\lambda) = \underbrace{\#_{(0,n]}(\psi_{0,n,0}(\lambda), \varphi_{1,n}(\lambda))}_{\geqslant N_1(\lambda)} = \underbrace{\#_{(0,n]}(\varphi_{0,n}(\lambda), \psi_{1,n,0}(\lambda))}_{\geqslant N_0(\lambda)}.$$

□

With respect to the following remark confer also Remark 8.22, Lemma 10.16, and Lemma 10.17.

Remark 10.4. *It is possible that we have $C_{0,z}(\lambda) > N_0(\lambda)$. Consider therefore the following example: let $\lambda \in \sigma_d(H_+^0)$, then*

$$W(u_{0,+}(\lambda), u_{0,-}(\lambda)) \text{ vanishes, thus } N_0(\lambda) = -1.$$

Let $z = \lambda + \varepsilon$, $\varepsilon > 0$, such that $[\lambda, z] \cap \sigma(H_+^0) = \{\lambda\}$. Then,

$$W(u_{0,+}(z), u_{0,-}(z)) \text{ is constant and nonzero, hence } C_{0,z}(\lambda) = N_0(z) = 0.$$

The same holds for $z = \lambda - \varepsilon$, where $\varepsilon > 0$ so that $[z, \lambda] \cap \sigma(H_+^0) = \{\lambda\}$.

Hence, by approximating twice we finally obtained the following two inequalities:

Lemma 10.5. *Let $b_0 \downarrow b_1$ near ∞, $[z_-, z_+] \cap \sigma_{ess}(H_+^0) = \emptyset$, and $i, j = 0, 1$, then*

$$E_{(z_-,z_+)}(H_+^1) - E_{(z_-,z_+]}(H_+^0) \leqslant N_i(z_+) - N_j(z_-),$$
$$E_{[z_-,z_+)}(H_+^1) - E_{(z_-,z_+)}(H_+^0) \geqslant N_i(z_+) - N_j(z_-).$$

Proof. By Lemma 10.3 we have

$$C_{0,z_+}(z_-) = N_0(z_+) - E_{(z_-,z_+)}(H_+^1) + E_{(z_-,z_+]}(H_+^0) \geqslant N_j(z_-),$$
$$C_{0,z_-}(z_+) = N_0(z_-) + E_{[z_-,z_+)}(H_+^1) - E_{(z_-,z_+)}(H_+^0) \geqslant N_j(z_+),$$
$$C_{1,z_+}(z_-) = N_1(z_+) - E_{(z_-,z_+)}(H_+^1) + E_{(z_-,z_+]}(H_+^0) \geqslant N_j(z_-),$$
$$C_{1,z_-}(z_+) = N_1(z_-) + E_{[z_-,z_+)}(H_+^1) - E_{(z_-,z_+)}(H_+^0) \geqslant N_j(z_+),$$

where $j = 0, 1$. □

Now we can already prove a first part of Theorem 1.2.

Lemma 10.6. *Let $b_0 \downarrow b_1$ near ∞ and $z \notin \sigma_{ess}(H_+^0)$, then*

$$N_0(z) = N_1(z). \tag{10.4}$$

Proof. Let $z_- < z < z_+$ such that

$$z_\pm \in \rho(H_+^0) \cap \rho(H_+^1) \quad \text{and} \quad [z_-, z_+] \cap \sigma_{ess}(H_+^0) = \emptyset$$

holds. If $z \notin \sigma(H_+^0)$, then by Lemma 10.5 we have

$$E_{[z_-,z)}(H_+^1) - E_{(z_-,z]}(H_+^0) = N_0(z) - N_0(z_-) = N_1(z) - N_0(z_-),$$

hence $N_0(z) = N_1(z)$. If $z \notin \sigma(H_+^1)$, then by Lemma 10.5 we have

$$E_{[z,z_+)}(H_+^1) - E_{(z,z_+]}(H_+^0) = N_0(z_+) - N_0(z) = N_0(z_+) - N_1(z),$$

thus $N_0(z) = N_1(z)$. If $z \in \sigma(H_+^0) \cap \sigma(H_+^1)$, then $u_{0,-}(z) = u_{0,+}(z)$ and $u_{1,-}(z) = u_{1,+}(z)$ holds, hence $N_0(z) = N_1(z)$. □

This shows, that the following is well-defined.

Definition 10.7. *Let $b_0 \downarrow b_1$ near ∞ and $z \notin \sigma_{ess}(H_+^0)$, then*

$$N(z) = \#_{(0,\infty]}(u_{0,+}(z), u_{1,-}(z)) = \#_{(0,\infty]}(u_{0,-}(z), u_{1,+}(z)) \tag{10.5}$$

holds.

From Lemma 10.5 and Lemma 10.6 we conclude the following corollary, which constitutes a first version of Theorem 1.2, but with the additional assumption (1.20).

Corollary 10.8. *Let $b_0 \downarrow b_1$ near ∞ and let $[z_-, z_+] \cap \sigma_{ess}(H^0_+) = \emptyset$. If $z_- \notin \sigma(H^1_+)$ and $z_+ \notin \sigma(H^0_+)$, then*

$$E_{[z_-,z_+)}(H^1_+) - E_{(z_-,z_+]}(H^0_+) = N(z_+) - N(z_-). \tag{10.6}$$

With respect to the continuous case the following should be mentioned:

Remark 10.9. *In the Sturm–Liouville case (Theorem 3.13 in [30]) and in the Dirac case (Theorem 1.1 in [37]) the assumption*

$$\lambda_0 \notin \sigma(H_1) \quad \text{and} \quad \lambda_1 \notin \sigma(H_0) \tag{10.7}$$

is missing for the equation, which holds in gaps of the essential spectrum above its infimum, i.e., in (3.10) and (1.12), respectively.

10.2 Main theorem for semi-infinite operators

To finally obtain Theorem 1.2 it remains to eliminate the assumption

$$z_- \notin \sigma(H^1_+) \quad \text{and} \quad z_+ \notin \sigma(H^0_+) \tag{10.8}$$

from Corollary 10.8, what is done in the present section.
As already mentioned in the introduction, see (1.20), we use a perturbation argument. With respect to the following lemma we remark, that a standard result of regular perturbation theory (confer the Kato–Rellich Theorem [34, Theorem XII.8]) also tells us, that the discrete eigenvalues of H^ε_+ are analytic functions of ε near $\varepsilon = 0$. Nonetheless, we prefer to give a self-contained proof for the following lemma, which follows from Lemma 10.5.

Lemma 10.10. *Let $z_- < z < z_+$, $z \in \sigma_d(H_+)$, $[z_-, z_+] \cap \sigma(H_+) = \{z\}$, and*

$$H^\varepsilon_+ = \begin{pmatrix} b(1)+\varepsilon & a(1) & \\ a(1) & b(2) & \ddots \\ & \ddots & \ddots \end{pmatrix}. \tag{10.9}$$

If $a(0)u_+(z_\pm, 0)$ and $a(0)u_+(z_\pm, 0) - \varepsilon u_+(z_\pm, 1) = a(0)u_{\varepsilon,+}(z_\pm, 0)$ are of the same sign (and non-zero), then

$$E_{[z_-,z_+]}(H^\varepsilon_+) = E_{(z_-,z_+)}(H^\varepsilon_+) = 1. \tag{10.10}$$

Moreover, $E_{(z,z_+)}(H_+^\varepsilon) = 1$ *if* $\varepsilon > 0$ *and* $E_{(z_-,z)}(H_+^\varepsilon) = 1$ *if* $\varepsilon < 0$.

Proof. Let $\varepsilon \neq 0$ and let all solutions be normalized such that either $u(1) = 1$ or $u(1) = 0$ holds. By $z \in \sigma_d(H_+)$ we have $u_+(z) = u_-(z)$ and $u_+(z, 0) = 0$. The difference equations τ and τ_ε coincide above $b(1)$, hence the solution $u_{\varepsilon,+}(z, j)$ of $(\tau_\varepsilon - z)u = 0$ which is square summable near ∞ coincides with $u_+(z, j)$ at $j \geqslant 1$. Then, $W_j(u_{\varepsilon,+}(z), u_-(z)) = W_j(u_+(z), u_-(z)) = 0$ for all $j \geqslant 1$ and $-a(0)u_{\varepsilon,+}(z, 0) = (b_0(1) + \varepsilon - z)u_{\varepsilon,+}(z, 1) + a(1)u_{\varepsilon,+}(z, 2) = \varepsilon u_+(z, 1) - a(0)u_+(z, 0) = \varepsilon \neq 0$, thus, $z \notin \sigma(H_+^\varepsilon)$, and $W_0(u_{\varepsilon,+}(z), u_-(z)) = a(0)u_{\varepsilon,+}(z, 0) = -\varepsilon$. Hence,

$$\begin{aligned}
&\#_{(0,\infty]}(u_{\varepsilon,+}(z), u_-(z)) \\
&\quad = \#_0(u_{\varepsilon,+}(z), u_-(z)) = \begin{cases} -1 = \#_{(0,\infty]}(u_+(z), u_-(z)) & \text{if } \varepsilon < 0 \\ 0 = \#_{(0,\infty]}(u_+(z), u_-(z)) + 1 & \text{if } \varepsilon > 0. \end{cases}
\end{aligned}$$

Further, the solutions $u_{\varepsilon,+}(z_\pm, j)$ coincide with $u_+(z_\pm, j)$ at $j \geqslant 1$, and hence $W_j(u_{\varepsilon,+}(z_\pm), u_-(z_\pm)) = W_j(u_+(z_\pm), u_-(z_\pm)) = W_0(u_+(z_\pm), u_-(z_\pm))$ holds for all $j \geqslant 1$ and moreover we have $W_1(u_{\varepsilon,+}(z_\pm), u_-(z_\pm)) = a(0)u_+(z_\pm, 0) \neq 0$, and $W_0(u_{\varepsilon,+}(z_\pm), u_-(z_\pm)) = a(0)u_+(z_\pm, 0) - \varepsilon u_+(z_\pm, 1)u_-(z_\pm, 1)$. Hence, $\#_0(u_{\varepsilon,+}(z_\pm), u_-(z_\pm)) = 0$ holds, that is, we have $\#_{(0,\infty]}(u_{\varepsilon,+}(z_\pm), u_-(z_\pm)) = 0$ if $a(0)u_+(z_\pm, 0)$ and $a(0)u_+(z_\pm, 0) - \varepsilon u_+(z_\pm, 1) = a(0)u_{\varepsilon,+}(z_\pm, 0)$ both are positive or both are negative. If so, then by Lemma 10.5 and $z_+ \notin \sigma(H_+^\varepsilon)$ we have

$$\begin{aligned}
1 - E_{(z_-,z_+]}(H_+^\varepsilon) &= E_{[z_-,z_+)}(H_+) - E_{(z_-,z_+]}(H_+^\varepsilon) \\
&= \#_{(0,\infty]}(u_{\varepsilon,+}(z_+), u_-(z_+)) - \#_{(0,\infty]}(u_{\varepsilon,+}(z_-), u_-(z_-)) = 0
\end{aligned}$$

hence, $E_{(z_-,z_+)}(H_+^\varepsilon) = E_{(z_-,z_+]}(H_+^\varepsilon) = 1$. Moreover, by Lemma 10.5

$$\begin{aligned}
-\, E_{(z_-,z]}(H_+^\varepsilon) &= E_{[z_-,z)}(H_+) - E_{(z_-,z]}(H_+^\varepsilon) \\
&= \#_{(0,\infty]}(u_{\varepsilon,+}(z), u_-(z)) - \#_{(0,\infty]}(u_{\varepsilon,+}(z_-), u_-(z_-)) \\
&= \#_{(0,\infty]}(u_+(z), u_-(z)) - \#_{(0,\infty]}(u_+(z_-), u_-(z_-)) + \begin{cases} 0 = -1 & \text{if } \varepsilon < 0 \\ 1 = 0 & \text{if } \varepsilon > 0 \end{cases}
\end{aligned}$$

holds, hence $E_{(z,z_+)}(H_+^\varepsilon) = 1$ if $\varepsilon > 0$ and $E_{(z_-,z)}(H_+^\varepsilon) = 1$ if $\varepsilon < 0$. □

Corollary 10.11. *The discrete spectrum of* H_+^ε *strictly increases (decreases) as* ε *increases (decreases). A point of* $\sigma_d(H_+^\varepsilon)$ *reaches the next point of* $\sigma(H_{1,+})$ *(if any) at* $\varepsilon = \infty$.

In the following lemma we consider the case, where one endpoint of the spectral interval is an eigenvalue of both operators.

Lemma 10.12. *Let $b_0 \downarrow b_1$ near ∞, $z_- < z < z_+$, $z \in \sigma_d(H_+^j)$, and $[z_-, z_+] \cap \sigma(H_+^j) = \{z\}$, $j = 0, 1$, then*

$$E_{[z_-,z)}(H_+^1) - E_{(z_-,z]}(H_+^0) = N(z) - N(z_-), \tag{10.11}$$
$$E_{[z,z_+)}(H_+^1) - E_{(z,z_+]}(H_+^0) = N(z_+) - N(z). \tag{10.12}$$

Proof. Let all solutions be normalized such that either $u(1) = 1$ or $u(1) = 0$ holds and let $\tilde{\tau}_0$ be the Jacobi difference equation corresponding to

$$\tilde{H}_+^0 = \begin{pmatrix} b_0(1) + \varepsilon & a(1) & \\ a(1) & b_0(2) & \ddots \\ & \ddots & \ddots \end{pmatrix}, \tag{10.13}$$

where $\varepsilon > 0$ is sufficiently small, then $E_{[z_-,z_+]}(\tilde{H}_+^0) = E_{(z,z_+)}(\tilde{H}_+^0) = 1$ holds by Lemma 10.10. The solutions $\tilde{u}_+$ and u_+ coincide at all points $j \geqslant 1$. Now, by $W_1(u_{0,+}(z), u_{1,-}(z)) = (b_0 - b_1)(1) u_{0,+}(z,1) u_{1,-}(z,1) = b_0(1) - b_1(1)$,

$$\begin{aligned} &W_0(\tilde{u}_{0,+}(z), u_{1,-}(z)) \\ &\quad = W_1(\tilde{u}_{0,+}(z), u_{1,-}(z)) - ((b_0 - b_1)(1) + \varepsilon)\tilde{u}_{0,+}(z,1) u_{1,-}(z,1) = -\varepsilon < 0, \end{aligned}$$

and $W_0(u_{0,+}(z), u_{1,-}(z)) = 0$ we have

$$\#_0(u_{0,+}(z), u_{1,-}(z)) = \#_0(\tilde{u}_{0,+}(z), u_{1,-}(z)) = \begin{cases} 0 & \text{if } (b_0 - b_1)(1) \leqslant 0 \\ 1 & \text{if } (b_0 - b_1)(1) > 0, \end{cases}$$

hence

$$\begin{aligned} N(z) &= \sum_{j=0}^{\infty} \#_j(u_{0,+}(z), u_{1,-}(z)) - 1 \\ &= \sum_{j=0}^{\infty} \#_j(\tilde{u}_{0,+}(z), u_{1,-}(z)) - 1 = \#_{(0,\infty]}(\tilde{u}_{0,+}(z), u_{1,-}(z)) - 1. \end{aligned}$$

We have $\#_j(\tilde{u}_{0,+}(z_-), u_{1,-}(z_-)) = \#_j(u_{0,+}(z_-), u_{1,-}(z_-))$ for all $j \geqslant 1$ and moreover $W_0(u_{0,+}(z_-), u_{1,-}(z_-)) \neq 0$ and $W_0(\tilde{u}_{0,+}(z_-), u_{1,-}(z_-)) \neq 0$ holds by $z_- \in \rho(H_+^0) \cap \rho(\tilde{H}_+^0)$. Further,

$$\begin{aligned} &W_1(u_{0,+}(z_-), u_{1,-}(z_-)) - W_0(u_{0,+}(z_-), u_{1,-}(z_-)) \\ &\quad = (b_0 - b_1)(1) u_{0,+}(z_-,1) u_{1,-}(z_-,1) = \begin{cases} 0 & \text{if } u_{0,+}(z_-,1) = 0 \\ (b_0 - b_1)(1) & \text{if } u_{0,+}(z_-,1) = 1 \end{cases} \end{aligned}$$

and

$$\begin{aligned}
&W_1(\tilde{u}_{0,+}(z_-), u_{1,-}(z_-)) - W_0(\tilde{u}_{0,+}(z_-), u_{1,-}(z_-)) \\
&\quad = ((b_0 - b_1)(1) + \varepsilon)\tilde{u}_{0,+}(z_-, 1)u_{1,-}(z_-, 1) \\
&\quad = \begin{cases} 0 & \text{if } u_{0,+}(z_-, 1) = 0 \\ (b_0 - b_1)(1) + \varepsilon & \text{if } u_{0,+}(z_-, 1) = 1. \end{cases}
\end{aligned}$$

If $u_{0,+}(z_-, 1) = 0$, then $\#_0(u_{0,+}(z_-), u_{1,-}(z_-)) = \#_0(\tilde{u}_{0,+}(z_-), u_{1,-}(z_-)) = 0$. If $u_{0,+}(z_-, 1) = 1$, then

$$\begin{aligned}
W_0(\tilde{u}_{0,+}(z_-), u_{1,-}(z_-)) &= W_1(u_{0,+}(z_-), u_{1,-}(z_-)) - b_0(1) + b_1(1) - \varepsilon \\
&= \underbrace{W_0(u_{0,+}(z_-), u_{1,-}(z_-))}_{\neq 0} - \varepsilon.
\end{aligned}$$

Now, let $\varepsilon > 0$ such that $W_0(\tilde{u}_{0,+}(z_-), u_{1,-}(z_-))$ and $W_0(u_{0,+}(z_-), u_{1,-}(z_-))$ are of the same sign. If $(b_0 - b_1)(1) = 0$, then by $W_1(\tilde{u}_{0,+}(z_-), u_{1,-}(z_-)) = W_1(u_{0,+}(z_-), u_{1,-}(z_-)) = W_0(u_{0,+}(z_-), u_{1,-}(z_-))$ we have

$$\#_0(u_{0,+}(z_-), u_{1,-}(z_-)) = \#_0(\tilde{u}_{0,+}(z_-), u_{1,-}(z_-)) = 0$$

If $(b_0 - b_1)(1) \neq 0$, then $\#_0(u_{0,+}(z_-), u_{1,-}(z_-)) = \#_0(\tilde{u}_{0,+}(z_-), u_{1,-}(z_-))$ if we choose $\varepsilon > 0$ sufficiently small such that moreover $(b_0 - b_1)(1)$ and $(b_0 - b_1)(1) + \varepsilon$ are of the same sign. Finally, in either case we have

$$\begin{aligned}
N(z_-) &= \sum_{j=0}^{\infty} \#_j(u_{0,+}(z_-), u_{1,-}(z_-)) \\
&= \sum_{j=0}^{\infty} \#_j(\tilde{u}_{0,+}(z_-), u_{1,-}(z_-)) = \#_{(0,\infty]}(\tilde{u}_{0,+}(z_-), u_{1,-}(z_-))
\end{aligned}$$

and thus by Lemma 10.5 we have

$$\begin{aligned}
&E_{[z_-,z)}(H_+^1) - E_{(z_-,z]}(H_+^0) = E_{[z_-,z)}(H_+^1) - E_{(z_-,z]}(\tilde{H}_+^0) - 1 \\
&\quad = \#_{(0,\infty]}(\tilde{u}_{0,+}(z), u_{1,-}(z)) - \#_{(0,\infty]}(\tilde{u}_{0,+}(z_-), u_{1,-}(z_-)) - 1 \\
&\quad = N(z) - N(z_-).
\end{aligned}$$

This proves the first claim. By Lemma 10.5

$$\begin{aligned}
&N(z_+) - N(z_-) = E_{[z_-,z_+)}(H_+^1) - E_{(z_-,z_+]}(H_+^0) \\
&\quad = E_{[z_-,z)}(H_+^1) - E_{(z_-,z]}(H_+^0) + E_{[z,z_+)}(H_+^1) - E_{(z,z_+]}(H_+^0) \\
&\quad = N(z) - N(z_-) + E_{[z,z_+)}(H_+^1) - E_{(z,z_+]}(H_+^0)
\end{aligned}$$

holds, which proves the second claim. □

Now we complete the proof of Theorem 1.2:

Theorem 10.13 (Relative oscillation theorem for semi-infinite Jacobi operators)**.** *Let* $[z_-, z_+] \cap \sigma_{ess}(H_+^0) = \emptyset$ *and* $b_0 \downarrow b_1$ *near* ∞*, then*

$$E_{[z_-,z_+)}(H_+^1) - E_{(z_-,z_+]}(H_+^0) = N(z_+) - N(z_-). \tag{10.14}$$

Proof of Theorem 10.13 and (1.17). Let $\varepsilon_+ > 0$ be sufficiently small such that $[z_+ - \varepsilon_+, z_+ + \varepsilon_+] \cap (\sigma(H_+^0) \cup \sigma(H_+^1)) \subseteq \{z_+\}$ and let $\alpha = z_+ - \varepsilon_+, \beta = z_+ + \varepsilon_+$. If $z_+ \in \sigma(H_+^0) \cap \sigma(H_+^1)$ or $z_+ \notin \sigma(H_+^0)$, then by Lemma 10.12 and Lemma 10.5 we have $E_{[\alpha,z_+)}(H_+^1) - E_{(\alpha,z_+]}(H_+^0) = N(z_+) - N(\alpha)$. If $z_+ \notin \sigma(H_+^1)$, then by Lemma 10.5 $E_{[z_+,\beta)}(H_+^1) - E_{(z_+,\beta]}(H_+^0) = N(\beta) - N(z_+)$ holds and hence by $E_{[\alpha,\beta)}(H_+^1) - E_{(\alpha,\beta]}(H_+^0) = N(\beta) - N(\alpha)$ we have

$$\begin{aligned} &E_{[\alpha,z_+)}(H_+^1) - E_{(\alpha,z_+]}(H_+^0) \\ &\quad = E_{[\alpha,\beta)}(H_+^1) - E_{(\alpha,\beta]}(H_+^0) - (E_{[z_+,\beta)}(H_+^1) - E_{(z_+,\beta]}(H_+^0)) \\ &\quad = N_0(\beta) - N(\alpha) - (N_0(\beta) - N(z_+)) = N(z_+) - N(\alpha). \end{aligned}$$

Let $\varepsilon_- > 0$ so that $[z_- - \varepsilon_-, z_- + \varepsilon_-] \cap (\sigma(H_+^0) \cup \sigma(H_+^1)) \subseteq \{z_-\}$ and let $\gamma = z_- - \varepsilon_-$ and $\delta = z_- + \varepsilon_-$.
If $z_- \in \sigma(H_+^0) \cap \sigma(H_+^1)$ or $z_- \notin \sigma(H_+^1)$, then by Lemma 10.12 and Lemma 10.5 we have $E_{[z_-,\delta)}(H_+^1) - E_{(z_-,\delta]}(H_+^0) = N(\delta) - N(z_-)$. If $z_- \notin \sigma(H_+^0)$, then by Lemma 10.5 $E_{[\gamma,z_-)}(H_+^1) - E_{(\gamma,z_-]}(H_+^0) = N(z_-) - N(\gamma)$ holds and hence by $E_{[\gamma,\delta)}(H_+^1) - E_{(\gamma,\delta]}(H_+^0) = N(\delta) - N(\gamma)$ we have

$$\begin{aligned} &E_{[z_-,\delta)}(H_+^1) - E_{(z_-\delta]}(H_+^0) \\ &\quad = N(\delta) - N(\gamma) - (N(z_-) - N(\gamma)) = N(\delta) - N(z_-). \end{aligned}$$

By Lemma 10.5 we have $E_{[\delta,\alpha)}(H_+^1) - E_{(\delta,\alpha]}(H_+^0) = N(\alpha) - N(\delta)$ and thus,

$$\begin{aligned} &E_{[z_-,z_+)}(H_+^1) - E_{(z_-,z_+]}(H_+^0) \\ &\quad = E_{[z_-,\delta)}(H_+^1) - E_{(z_-\delta]}(H_+^0) + E_{[\delta,\alpha)}(H_+^1) - E_{(\delta,\alpha]}(H_+^0) \\ &\qquad + E_{[\alpha,z_+)}(H_+^1) - E_{(\alpha,z_+]}(H_+^0) \\ &\quad = N(\delta) - N(z_-) + N(\alpha) - N(\delta) + N(z_+) - N(\alpha) = N(z_+) - N(z_-). \end{aligned}$$

□

Corollary 10.14. *Let* I *be a connected component of* $\mathbb{R} \setminus \sigma_{ess}(H_+^0)$*. Then the function*

$$N : I \subseteq \mathbb{R} \setminus \sigma_{ess}(H_+^0) \to \mathbb{Z} \tag{10.15}$$

from (10.5) is a step function which jumps (left-continuous) by 1 *at each eigenvalue of* H^1_+ *and jumps (right-continuous) by* -1 *at each eigenvalue of* H^0_+*. The function* N *is continuous at all* z *in both resolvent sets and jumps locally by* -1 *at all* z *in both spectra, that is, we then have* $N(z-\varepsilon) = N(z) + 1 = N(z+\varepsilon)$ *for all* ε *sufficiently small.*

Finally, we want to have a closer look at the approximation again, thereto we add the following claim.

Lemma 10.15. *Let* $\lambda_j \notin \sigma_{ess}(H^j_+)$ *and let* $u_{j,+}(\lambda_j), j = 0, 1,$ *be Weyl solutions of* $(\tau_j - \lambda_j)u_j = 0$*. Then,*

$$\mathscr{J}_{u_{0,+}(\lambda_0)} \cap \mathscr{J}_{u_{1,+}(\lambda_1)}$$

is an infinite set. The same holds for solutions which are square summable near $-\infty$*.*

Proof. Abbreviate $u_j = u_{j,+}(\lambda_j), j = 0, 1,$ and suppose J is a finite set. Then, since the nodes of both solutions are simple, without loss, there exists an $N \in \mathbb{N}$ such that $u_0(k) = 0$ for all $k > N, k$ even, and $u_1(k) = 0$ for all $k > N, k$ odd. If so, by

$$W_{n+1}(u_0, u_1) - W_n(u_0, u_1) = (b_0 - b_1)(n+1)u_0(n+1)u_1(n+1) = 0$$

the Wronskian is constant near ∞. Moreover, the Wronskian is not vanishing near ∞ by $W_n(u_0, u_1) = a(n)(u_0(n)u_1(n+1) - u_1(n)u_0(n+1)) \neq 0$ and thus $W(u_0, u_1) \notin \ell^2(\mathbb{N})$ which contradicts Lemma 3.6. □

By Lemma 10.3 and Theorem 10.13 we obtain the following lemma, which shows explicitly, for which boundary conditions the Wronskians associated with the finite matrices actually have one node more than the semi-infinite one – although we have convergence on an (arbitrary) finite set.

Lemma 10.16. *Let* $b_0 \downarrow b_1$ *near* ∞*,* $\lambda, z \in [z_-, z_+] \cap \sigma_{ess}(H^0_+) = \emptyset$*, and* $v_0 = u_{0,+}(z), v_1 = u_{1,+}(z)$*. Then, for all* $\lambda \in [z_-, z_+]$ *there exists an* N_λ *such that*

$$\begin{aligned} E_{(-\infty,\lambda)}(H^{1,v_0}_n) - E_{(-\infty,\lambda]}(H^{0,v_0}_n) &= C_{0,z}(\lambda) \\ = E_{(-\infty,\lambda)}(H^{1,v_1}_n) - E_{(-\infty,\lambda]}(H^{0,v_1}_n) &= C_{1,z}(\lambda) \\ = \begin{cases} N(\lambda) + 1 & \text{if } z < \lambda \in \sigma(H^0_+) \text{ or } z > \lambda \in \sigma(H^1_+) \\ N(\lambda) & \text{otherwise} \end{cases} \end{aligned}$$

for all $n \geqslant N_\lambda$*,* $n \in \mathscr{J}_{v_0} \cap \mathscr{J}_{v_1}$*, which is an infinite set by Lemma 10.15.*

Proof. By Lemma 10.3 we have

$$C_{0,z}(\lambda) = C_{1,z}(\lambda) = N(z) + \begin{cases} E_{[z,\lambda)}(H_+^1) - E_{(z,\lambda)}(H_+^0) & \text{if } \lambda > z \\ 0 & \text{if } \lambda = z \\ -E_{(\lambda,z)}(H_+^1) + E_{(\lambda,z]}(H_+^0) & \text{if } \lambda < z. \end{cases}$$

Let $j = 0, 1$. If $\lambda > z$, then by Theorem 10.13 we have

$$\begin{aligned} N(\lambda) - N(z) &= E_{[z,\lambda)}(H_+^1) - E_{(z,\lambda]}(H_+^0) \\ &= E_{[z,\lambda)}(H_+^1) - E_{(z,\lambda)}(H_+^0) - \begin{cases} 1 & \text{if } \lambda \in \sigma(H_+^0) \\ 0 & \text{otherwise} \end{cases} \\ &= C_{j,z}(\lambda) - N(z) - \begin{cases} 1 & \text{if } \lambda \in \sigma(H_+^0) \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

thus, $C_{j,z}(\lambda) = N(\lambda) + 1$ if $\lambda \in \sigma(H_+^0)$. If $\lambda < z$, then by Theorem 10.13 we have

$$\begin{aligned} N(z) - N(\lambda) &= E_{[\lambda,z)}(H_+^1) - E_{(\lambda,z]}(H_+^0) \\ &= N(z) - C_{j,z}(\lambda) + \begin{cases} 1 & \text{if } \lambda \in \sigma(H_+^1) \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

thus, $C_{j,z}(\lambda) = N(\lambda) + 1$ if $\lambda \in \sigma(H_+^1)$. □

Hence, now we see explicitly, that an eigenvalue at the 'foreign' closed endpoint of the spectral interval is approximated from outside the interval. Thereto, confer also Remark 8.22, Remark 10.4, and Lemma 10.16.

Lemma 10.17. *Let $b_0 \downarrow b_1$ near ∞ and $[z_-, z_+] \cap \sigma_{ess}(H_+^0) = \emptyset$. Then, there exists an N such that*

$$E_{[z_-,z_+)}(H_n^{1,v}) - E_{(z_-,z_+]}(H_n^{0,v}) = N(z_+) - N(z_-) - \begin{cases} 1 & \text{if } z_- \in \sigma(H_+^1) \\ 0 & \text{otherwise} \end{cases}$$

if $v = u_{0,+}(z_+)$ or $v = u_{1,+}(z_+)$ and

$$E_{[z_-,z_+)}(H_n^{1,v}) - E_{(z_-,z_+]}(H_n^{0,v}) = N(z_+) - N(z_-) + \begin{cases} 1 & \text{if } z_+ \in \sigma(H_+^0) \\ 0 & \text{otherwise} \end{cases}$$

if $v = u_{0,+}(z_-)$ or $v = u_{1,+}(z_-)$ holds for all $n \in \mathscr{J}_v$, where $n \geqslant N$.

Proof. Let $j = 0, 1$. If $v_j = u_{j,+}(z_+)$, then by Lemma 10.16 we have

$$E_{(-\infty,z_+)}(H_n^{1,v_j}) - E_{(-\infty,z_+]}(H_n^{0,v_j}) = N(z_+),$$
$$E_{(-\infty,z_-)}(H_n^{1,v_j}) - E_{(-\infty,z_-]}(H_n^{0,v_j}) = N(z_-) + \begin{cases} 1 & \text{if } z_- \in \sigma(H_+^1) \\ 0 & \text{otherwise} \end{cases}$$

holds for all $n \in \mathscr{J}_{v_j}$ sufficiently large. If $v_j = u_{j,+}(z_-)$, then by Lemma 10.16 we have

$$E_{(-\infty,z_-)}(H_n^{1,v_j}) - E_{(-\infty,z_-]}(H_n^{0,v_j}) = N(z_-),$$
$$E_{(-\infty,z_+)}(H_n^{1,v_j}) - E_{(-\infty,z_+]}(H_n^{0,v_j}) = N(z_+) + \begin{cases} 1 & \text{if } z_+ \in \sigma(H_+^0) \\ 0 & \text{otherwise.} \end{cases}$$

□

10.3 A proof for finite-rank perturbations

We want to remark that, if the perturbation $b_0 - b_1$ is finite rank, then Theorem 10.13 can be obtained more easily from Corollary 10.8. That is, for the finite-rank case Section 10.2 can be replaced by this one.

Lemma 10.18. *If $W(u_{0,+}(z), u_{1,-}(z)) \equiv 0$ near ∞, then $z \in \sigma(H_+^1)$.*

Proof. By Lemma 3.7 both solutions are linearly dependent near ∞ and hence $u_{1,-}(z)$ is square summable near ∞, thus $z \in \sigma(H_+^1)$. □

Lemma 10.19. *Let $z, \lambda \in [\lambda_0, \lambda_1] \cap \sigma_{ess}(H_+^0) = \emptyset$. If $\dim \operatorname{Ran}(H_+^0 - H_+^1) < \infty$ and $\lambda \in \rho(H_+^0) \cap \rho(H_+^1)$, then*

$$C_{0,z}(\lambda) = N_0(\lambda).$$

Proof. We have $N_0(\lambda) < \infty$ by Theorem 7.10. Moreover, $W(u_{0,+}(\lambda), u_{1,-}(\lambda))$ is either positive or negative near ∞ by Lemma 10.18 and Lemma 7.4. Let N, such that $(b_0 - b_1)(j) = 0$ and $W_j(u_{0,+}(\lambda), u_{1,-}(\lambda)) \neq 0$ for all $j \geqslant N$. Then, $W_j(u_{0,+}(\lambda), u_{1,-}(\lambda)) = c$ is constant for all $j \geqslant N - 1$ by

$$\begin{aligned} &W_{j+1}(u_{0,+}(\lambda), u_{1,-}(\lambda)) - W_j(u_{0,+}(\lambda), u_{1,-}(\lambda)) \\ &\quad = (b_0 - b_1)(j+1) u_{0,+}(\lambda, j+1) u_{1,-}(\lambda, j+1) = 0 \end{aligned}$$

and hence all nodes of $W(u_{0,+}(\lambda), u_{1,-}(\lambda))$ are to the left of N.
Let $n \in \mathscr{J}_v, v = u_{0,+}(z)$, and let $\phi_{j,n}(\lambda)$, $j = 0, 1$, be any solutions of $(\tau_{j,n} - \lambda)u = 0$, then, $W_j(\phi_{0,n}(\lambda), \phi_{1,n}(\lambda)) = \tilde{c}$ is constant for all $j \geqslant N - 1$.

By Lemma 8.21 there exist solutions $\varphi_{0,n}(\lambda)$ of $(\tau_{0,n} - \lambda)u = 0$ such that $\varphi_{0,n}(\lambda, n) = 0$ and $\varphi_{0,n}(\lambda, m) \to u_{0,+}(\lambda, m)$ at $m = -1, \dots, N$. We have $\#_{(0,N-1]}(\varphi_{0,n}(\lambda), u_{1,-}(\lambda)) = \#_{(0,N-1]}(u_{0,+}(\lambda), u_{1,-}(\lambda))$ by Lemma 8.26 and $W_0(u_{0,+}(\lambda), u_{1,-}(\lambda)) \neq 0$, and thus

$$\begin{aligned} C_{0,z}(\lambda) &= E_{(-\infty,\lambda)}(H_n^{1,v}) - E_{(-\infty,\lambda]}(H_n^{0,z}) = \#_{(0,n]}(\psi_{0,n,n}(\lambda), \psi_{1,n,0}(\lambda)) \\ &= \#_{(0,N-1]}(\psi_{0,n,n}(\lambda), \psi_{1,n,0}(\lambda)) = \#_{(0,N-1]}(\varphi_{0,n}(\lambda), u_{1,-}(\lambda)) = N_0(\lambda) \end{aligned}$$

holds for all n sufficiently large. □

Lemma 10.20. *Let* $[z_-, z_+] \cap \sigma_{ess}(H_+^0) = \emptyset$. *If* $\dim \operatorname{Ran}(H_+^0 - H_+^1) < \infty$, *then*

$$E_{[z_-,z_+)}(H_+^1) - E_{(z_-,z_+]}(H_+^0) = N_0(z_+) - N_0(z_-).$$

Proof. Let $\lambda_- < \lambda_+, \lambda_\pm \in (z_-, z_+)$ such that $\lambda_\pm \in \rho(H_+^0) \cap \rho(H_+^1)$. Then, by Lemma 10.19 and Lemma 10.3 we have

$$\begin{aligned} C_{0,z_-}(\lambda_-) &= N_0(\lambda_-) = N_0(z_-) + E_{[z_-,\lambda_-)}(H_+^1) - E_{(z_-,\lambda_-)}(H_+^0), \\ C_{0,z_+}(\lambda_+) &= N_0(\lambda_+) = N_0(z_+) - E_{(\lambda_+,z_+)}(H_+^1) + E_{(\lambda_+,z_+]}(H_+^0). \end{aligned}$$

Now,

$$\begin{aligned} &E_{[z_-,z_+)}(H_+^1) - E_{(z_-,z_+]}(H_+^0) \\ &\quad = E_{[z_-,\lambda_-)}(H_+^1) - E_{(z_-,\lambda_-]}(H_+^0) + E_{[\lambda_-,\lambda_+)}(H_+^1) \\ &\qquad - E_{(\lambda_-,\lambda_+]}(H_+^0) + E_{[\lambda_+,z_+)}(H_+^1) - E_{(\lambda_+,z_+]}(H_+^0) \\ &\quad = N_0(\lambda_-) - N_0(z_-) + N_0(\lambda_+) - N_0(\lambda_-) + N_0(z_+) - N_0(\lambda_+) \end{aligned}$$

holds by Lemma 10.5. □

Chapter 11

Infinite Jacobi operators

It remains to look at gaps of the essential spectrum of infinite Jacobi operators. This is done in the present chapter, where we in the end complete the proof of Theorem 1.1.

11.1 A first theorem on the line

From now on we use the notation from Subsection 8.2.2 again and remark, that u_- now denotes a solution fulfilling the left boundary condition of H, that is, $u_- \in \ell^2(-\mathbb{N})$. Moreover, we abbreviate

$$\mathcal{N}_0(z) = \#_{(-\infty,\infty]}(u_{0,+}(z), u_{1,-}(z)), \tag{11.1}$$

$$\mathcal{N}_1(z) = \#_{(-\infty,\infty]}(u_{0,-}(z), u_{1,+}(z)). \tag{11.2}$$

Since we are interested in the discrete spectrum of H we assume

$$[z_-, z_+] \cap \sigma_{ess}(H_0) = \emptyset, \tag{11.3}$$

$z_- < z_+$, and hence we also have $[z_-, z_+] \cap \sigma_{ess}(H^{0,w}_{m,+}) = \emptyset$ for all $m \in \mathscr{J}_w$. For all $z, \tilde{z} \in [z_-, z_+]$ we have $\tau_0 - z \overset{rno}{\sim} \tau_1 - \tilde{z}$ by Theorem 7.11, thus $\mathcal{N}_0(z)$ and $\mathcal{N}_1(z)$ are finite numbers.

First of all, we approximate the infinite operators by semi-infinite operators and compare their spectra as well as the number of nodes of the corresponding Wronskians, which is done in the following lemma:

Lemma 11.1. *Let $b_0 \downarrow b_1$ near $+\infty$ and near $-\infty$, and $z \in [z_-, z_+] \cap \sigma_{ess}(H_0) = \emptyset$. If $w = u_{1,-}(z)$, then for all $\lambda \in [z_-, z_+]$, $\lambda \neq z$, there exists an N_λ and a constant $\mathcal{C}_{0,z}(\lambda) \in \mathbb{Z}$ such that*

$$\mathcal{N}_0(\lambda) \leqslant \mathcal{C}_{0,z}(\lambda) = \#_{(m,\infty]}(\psi_{0,m,+}(\lambda), \psi_{1,m,m}(\lambda))$$

holds for all $m < N_\lambda$, $m \in \mathscr{J}_w$.
If $w = u_{0,-}(z)$*, then for all* $\lambda \in [z_-, z_+], \lambda \neq z$*, there exists an* N_λ *and a constant* $\mathcal{C}_{1,z}(\lambda) \in \mathbb{Z}$ *such that*

$$\mathcal{N}_1(\lambda) \leqslant \mathcal{C}_{1,z}(\lambda) = \#_{(m,\infty]}(\psi_{0,m,m}(\lambda), \psi_{1,m,+}(\lambda))$$

holds for all $m < N_\lambda$, $m \in \mathscr{J}_w$*. Moreover,*

$$\mathcal{C}_{0,z}(\lambda) - \mathcal{N}_0(z) = \mathcal{C}_{1,z}(\lambda) - \mathcal{N}_1(z) = \begin{cases} E_{[z,\lambda)}(H_1) - E_{(z,\lambda)}(H_0) & \text{if } \lambda > z \\ -E_{(\lambda,z)}(H_1) + E_{(\lambda,z]}(H_0) & \text{if } \lambda < z \end{cases}$$

and if $\lambda \neq z$*, then* $\mathcal{N}_1(\lambda) \leqslant \mathcal{C}_{0,z}(\lambda)$ *and* $\mathcal{N}_0(\lambda) \leqslant \mathcal{C}_{1,z}(\lambda)$.

Proof. Let all nodes of $W(u_{0,+}(z), u_{1,-}(z))$ and $W(u_{0,-}(z), u_{1,+}(z))$ be to the right of N. First, let $w = u_{1,-}(z)$, where $m \in \mathscr{J}_w$. If $\lambda < z$, then there exists an $N_\lambda < N$ such that by Lemma 8.16 and Corollary 8.9

$$E_{[\lambda,z)}(H^{1,z}_{m,+}) = E_{(\lambda,z)}(H^{1,z}_{m,+}) = E_{(\lambda,z)}(H_1) = M_1$$

holds and moreover by $z + b_0 - b_1 \downarrow z$ and Lemma 8.10 we have $E_{(\lambda,z]}(H^{0,w}_{m,+}) = E_{(\lambda,z]}(H_0) = M_0$ for all $m < N_\lambda$. Thus, by (1.17) and Lemma 8.20 we have

$$\begin{aligned} M_1 - M_0 &= E_{[\lambda,z)}(H^{1,z}_{m,+}) - E_{(\lambda,z]}(H^{0,w}_{m,+}) \\ &= \#_{(m,\infty]}(\psi_{0,m,+}(z), \psi_{1,m,m}(z)) - \#_{(m,\infty]}(\psi_{0,m,+}(\lambda), \psi_{1,m,m}(\lambda)) \\ &= \mathcal{N}_0(z) - \mathcal{C}_{0,z}(\lambda) \end{aligned} \tag{11.4}$$

for all $m < N_\lambda$. If $\lambda > z$, then there exists an $N_\lambda < N$ such that we have $E_{[z,\lambda)}(H^{1,z}_{m,+}) = E_{[z,\lambda)}(H_1) = \tilde{M}_1$ by Corollary 8.11 and moreover by Lemma 8.16, $z + b_0 - b_1 \downarrow z$, and Lemma 8.8

$$E_{(z,\lambda]}(H^{0,w}_{m,+}) = E_{(z,\lambda)}(H^{0,w}_{m,+}) = E_{(z,\lambda)}(H_0) = \tilde{M}_0$$

holds for all $m < N_\lambda$. Now, by (1.17) and Lemma 8.20 we have

$$\begin{aligned} \tilde{M}_1 - \tilde{M}_0 &= E_{[z,\lambda)}(H^{1,z}_{m,+}) - E_{(z,\lambda]}(H^{0,w}_{m,+}) \\ &= \#_{(m,\infty]}(\psi_{0,m,+}(\lambda), \psi_{1,m,m}(\lambda)) - \#_{(m,\infty]}(\psi_{0,m,+}(z), \psi_{1,m,m}(z)) \\ &= \mathcal{C}_{0,z}(\lambda) - \mathcal{N}_0(z) \end{aligned} \tag{11.5}$$

for all $m < N_\lambda$.
Now, let $w = u_{0,-}(z)$, where $m \in \mathscr{J}_w$. If $\lambda < z$, then there exists an $N_\lambda < N$ such that we have $E_{(\lambda,z]}(H^{0,z}_{m,+}) = E_{(\lambda,z]}(H_0) = M_0 < \infty$ by Corollary 8.11 and

moreover by Lemma 8.16 and by $z + b_1 - b_0 \uparrow z$ and Lemma 8.8

$$E_{[\lambda,z)}(H^{1,w}_{m,+}) = E_{(\lambda,z)}(H^{1,w}_{m,+}) = E_{(\lambda,z)}(H_1) = M_1 < \infty$$

holds for all $m < N_\lambda$. Thus, by (1.17) and Lemma 8.20 we have

$$\begin{aligned} M_1 - M_0 &= E_{[\lambda,z)}(H^{1,w}_{m,+}) - E_{(\lambda,z]}(H^{0,z}_{m,+}) \\ &= \#_{(m,\infty]}(\psi_{0,m,m}(z), \psi_{1,m,+}(z)) - \#_{(m,\infty]}(\psi_{0,m,m}(\lambda), \psi_{1,m,+}(\lambda)) \\ &= \mathcal{N}_1(z) - \mathcal{C}_{1,z}(\lambda) \end{aligned} \tag{11.6}$$

for all $m < N_\lambda$, $m \in \mathscr{J}_w$. If $\lambda > z$, then there exists an $N_\lambda < N$ such that by Lemma 8.16 and Corollary 8.9

$$E_{(z,\lambda]}(H^{0,z}_{m,+}) = E_{(z,\lambda)}(H^{0,z}_{m,+}) = E_{(z,\lambda)}(H_0) = \tilde{M}_0 < \infty$$

and moreover and by $z + b_1 - b_0 \uparrow z$ and Lemma 8.10 we have $E_{[z,\lambda)}(H^{1,w}_{m,+}) = E_{[z,\lambda)}(H_1) = \tilde{M}_1 < \infty$ for all $m < N_\lambda$. Now, by (1.17) and Lemma 8.20 we have

$$\begin{aligned} \tilde{M}_1 - \tilde{M}_0 &= E_{[z,\lambda)}(H^{1,w}_{m,+}) - E_{(z,\lambda]}(H^{0,z}_{m,+}) \\ &= \#_{(m,\infty]}(\psi_{0,m,m}(\lambda), \psi_{1,m,+}(\lambda)) - \#_{(m,\infty]}(\psi_{0,m,m}(z), \psi_{1,m,+}(z)) \\ &= \mathcal{C}_{1,z}(\lambda) - \mathcal{N}_1(z) \end{aligned} \tag{11.7}$$

for all $m < N_\lambda$, $m \in \mathscr{J}_w$.
For the remaining inequalities, note that in either case, if $\lambda \neq z$, there exist L, K such that $b_0(j) - b_1(j) \geqslant 0$ for all $j \leqslant L, j \geqslant K$, and moreover $W(u_{0,+}(\lambda), u_{1,-}(\lambda))$ as well as $W(u_{0,-}(\lambda), u_{1,+}(\lambda))$ is of one sign (or vanishing) for all $j \leqslant L$ and all $j \geqslant K$.
Let $j = 0, 1$, then by Lemma 8.23 there exist solutions $\varphi_{j,m}(\lambda)$ of $(\tau_{j,m} - \lambda)\varphi(\lambda) = 0$ such that $\varphi_{j,m}(\lambda, m) = 0$ and $\varphi_{j,m}(\lambda, n) \to u_{j,-}(\lambda, n)$ at $n = K-1, \ldots, L+1$. The solution $u_{j,+}(\lambda)$ is a solution of $(\tau_{j,m} - \lambda)u = 0$ above m. Moreover by Lemma 8.16 we have $\lambda \in \rho(H^{0,w}_{m,+}) \cap \rho(H^{1,w}_{m,+})$ for all $|m|$ sufficiently large, thus by Lemma 8.26 we have

$$\begin{aligned} \mathcal{C}_{0,z}(\lambda) = \#_{(m,\infty]}(\psi_{0,m,+}(\lambda), \varphi_{1,m}(\lambda)) &= \#_{[m,\infty]}(\psi_{0,m,+}(\lambda), \varphi_{1,m}(\lambda)) \\ \geqslant \#_{[K,L]}(\psi_{0,m,+}(\lambda), \varphi_{1,m}(\lambda)) &\geqslant \#_{(K,L]}(\psi_{0,m,+}(\lambda), \varphi_{1,m}(\lambda)) \\ = \#_{(K,L]}(u_{0,+}(\lambda), \varphi_{1,m}(\lambda)) &\geqslant \#_{(K,L]}(u_{0,+}(\lambda), u_{1,-}(\lambda)) = \mathcal{N}_0(\lambda) \end{aligned} \tag{11.8}$$

as well as

$$\begin{aligned} \mathcal{C}_{0,z}(\lambda) = \#_{(m,\infty]}(\varphi_{0,m}(\lambda), \psi_{1,m,+}(\lambda)) &= \#_{[m,\infty]}(\varphi_{0,m}(\lambda), \psi_{1,m,+}(\lambda)) \\ \geqslant \#_{[K,L]}(\varphi_{0,m}(\lambda), \psi_{1,m,+}(\lambda)) &\geqslant \#_{(K,L]}(\varphi_{0,m}(\lambda), \psi_{1,m,+}(\lambda)) \\ = \#_{(K,L]}(\varphi_{0,m}(\lambda), u_{1,+}(\lambda)) &\geqslant \#_{(K,L]}(u_{0,-}(\lambda), u_{1,+}(\lambda)) = \mathcal{N}_1(\lambda), \end{aligned} \tag{11.9}$$

if $w = u_{1,-}(z)$, and analogously

$$\begin{aligned} \mathcal{C}_{1,z}(\lambda) &= \#_{(m,\infty]}(\varphi_{0,m}, \psi_{1,m,+}(\lambda)) \geqslant \#_{(K,L]}(\varphi_{0,m}, \psi_{1,m,+}(\lambda)) \\ &= \#_{(K,L]}(\varphi_{0,m}(\lambda), u_{1,+}(\lambda)) \geqslant \#_{(K,L]}(u_{0,+}(\lambda), u_{1,-}(\lambda)) = \mathcal{N}_1(\lambda) \end{aligned} \tag{11.10}$$

and

$$\begin{aligned} \mathcal{C}_{1,z}(\lambda) &= \#_{(m,\infty]}([\psi_{0,m,+}(\lambda), \varphi_{1,m}(\lambda)) \geqslant \#_{(K,L]}(\psi_{0,m,+}(\lambda), \varphi_{1,m}(\lambda)) \\ &= \#_{(K,L]}(u_{0,+}(\lambda), \varphi_{1,m}(\lambda)) \geqslant \#_{(K,L]}(u_{0,+}(\lambda), u_{1,-}(\lambda)) = \mathcal{N}_0(\lambda) \end{aligned} \tag{11.11}$$

holds if $w = u_{0,-}(z)$. □

This leads to the infinite counterpart of the inequalities, which we've already obtained in the semi-infinite case:

Lemma 11.2. *Let $b_0 \downarrow b_1$ near $+\infty$ and near $-\infty$, $[z_-, z_+] \cap \sigma_{ess}(H_0) = \emptyset$, and $i, j = 0, 1$, then*

$$\begin{aligned} E_{(z_-,z_+)}(H_1) - E_{(z_-,z_+]}(H_0) &\leqslant \mathcal{N}_i(z_+) - \mathcal{N}_j(z_-), \\ E_{[z_-,z_+)}(H_1) - E_{(z_-,z_+)}(H_0) &\geqslant \mathcal{N}_i(z_+) - \mathcal{N}_j(z_-). \end{aligned}$$

Proof. By Lemma 11.1 we have

$$\mathcal{C}_{0,z_+}(z_-) = \mathcal{N}_0(z_+) - E_{(z_-,z_+)}(H_1) + E_{(z_-,z_+]}(H_0) \geqslant \mathcal{N}_j(z_-), \tag{11.12}$$

$$\mathcal{C}_{1,z_+}(z_-) = \mathcal{N}_1(z_+) - E_{(z_-,z_+)}(H_1) + E_{(z_-,z_+]}(H_0) \geqslant \mathcal{N}_j(z_-), \tag{11.13}$$

$$\mathcal{C}_{0,z_-}(z_+) = \mathcal{N}_0(z_-) + E_{[z_-,z_+)}(H_1) - E_{(z_-,z_+)}(H_0) \geqslant \mathcal{N}_j(z_+), \tag{11.14}$$

$$\mathcal{C}_{1,z_-}(z_+) = \mathcal{N}_1(z_-) + E_{[z_-,z_+)}(H_1) - E_{(z_-,z_+)}(H_0) \geqslant \mathcal{N}_j(z_+). \tag{11.15}$$

□

In the following lemma we now already obtain one part of Theorem 1.1.

Lemma 11.3. *Let $b_0 \downarrow b_1$ near $+\infty$ and near $-\infty$, $z \notin \sigma_{ess}(H_0)$, then*

$$\mathcal{N}_0(z) = \mathcal{N}_1(z). \tag{11.16}$$

Proof. Let $z_- < z < z_+$ so that

$$z_\pm \in \rho(H_0) \cap \rho(H_1) \quad \text{and} \quad [z_-, z_+] \cap \sigma_{ess}(H_0) = \emptyset$$

holds. If $z \notin \sigma(H_0)$, then by Lemma 11.2 we have

$$E_{[z_-,z)}(H_1) - E_{(z_-,z]}(H_0) = \mathcal{N}_0(z) - \mathcal{N}_0(z_-) = \mathcal{N}_1(z) - \mathcal{N}_0(z_-), \tag{11.17}$$

hence $\mathcal{N}_0(z) = \mathcal{N}_1(z)$. If $z \notin \sigma(H_1)$, then by Lemma 11.2 we have

$$E_{[z,z_+)}(H_1) - E_{(z,z_+]}(H_0) = \mathcal{N}_0(z_+) - \mathcal{N}_0(z) = \mathcal{N}_0(z_+) - \mathcal{N}_1(z), \qquad (11.18)$$

thus $\mathcal{N}_0(z) = \mathcal{N}_1(z)$. If we have $z \in \sigma(H_0) \cap \sigma(H_1)$, then $u_{0,-}(z) = u_{0,+}(z)$ and $u_{1,-}(z) = u_{1,+}(z)$ holds, hence $\mathcal{N}_0(z) = \mathcal{N}_1(z)$. □

Now, the following is well-defined:

Definition 11.4. *Let $b_0 \downarrow b_1$ near $+\infty$ and near $-\infty$, $z \notin \sigma_{ess}(H_0)$, then*

$$\mathcal{N}(z) = \#_{(-\infty,\infty]}(u_{0,+}(z), u_{1,-}(z)) = \#_{(-\infty,\infty]}(u_{0,-}(z), u_{1,+}(z)).$$

From Lemma 11.2 and Lemma 11.3 we now obtain a first version of Theorem 1.1, but with the additional assumption (1.21).

Corollary 11.5. *Let $[z_-, z_+] \cap \sigma_{ess}(H_0) = \emptyset$ and let $b_0 \downarrow b_1$ near $+\infty$ and near $-\infty$. If $z_- \notin \sigma(H_1)$ and $z_+ \notin \sigma(H_0)$, then*

$$E_{[z_-,z_+)}(H_1) - E_{(z_-,z_+]}(H_0) = \mathcal{N}(z_+) - \mathcal{N}(z_-).$$

With respect to the Sturm–Liouville and Dirac counterparts we refer to Remark 10.9. It remains to eliminate the assumption

$$z_- \notin \sigma(H_1) \quad \text{and} \quad z_+ \notin \sigma(H_0), \qquad (11.19)$$

what is done in the sequel.

11.2 The finite-rank case

First of all, we eliminate the assumption (1.21) for the case of finite-rank perturbations.

Lemma 11.6. *We have*

$$W(u_{0,+}(z), u_{1,-}(z)) \text{ vanishes near} \begin{cases} +\infty \implies z \in \sigma(H_1) \\ -\infty \implies z \in \sigma(H_0). \end{cases} \qquad (11.20)$$

Proof. If $W(u_{0,+}(z), u_{1,-}(z))$ vanishes near $+\infty$, then by Lemma 3.7 both solutions are linearly dependent near $+\infty$ and hence $u_{1,-}(z)$ is square summable near $+\infty$. Analogously near $-\infty$. □

Lemma 11.7. *Let $z, \lambda \in [z_-, z_+] \cap \sigma_{ess}(H_0) = \emptyset$, $z \neq \lambda$, and let moreover $\dim \operatorname{Ran}(H_0 - H_1) < \infty$ and $\lambda \in \rho(H_0) \cap \rho(H_1)$ hold, then*

$$\mathcal{C}_{0,z}(\lambda) = \mathcal{N}(\lambda).$$

Proof. By

$$\begin{aligned} &W_{n+1}(u_{0,+}(\lambda), u_{1,-}(\lambda)) - W_n(u_{0,+}(\lambda), u_{1,-}(\lambda)) \\ &\quad = \underbrace{(b_0 - b_1)(n+1)}_{=0} u_{0,+}(\lambda, n+1) u_{1,-}(\lambda, n+1) \end{aligned}$$

there exists an N such that the Wronskian is constant (and nonvanishing by Lemma 11.6) for all $j \leqslant -N$ and for all $j \geqslant N$. Thus, all nodes of the Wronskian are in $-N, \ldots, N-1$. By Lemma 8.23 for all $m \in \mathscr{J}_w$, where $w = u_{1,-}(z)$, there exist solutions $\varphi_m(\lambda)$ of $(\tau_{1,m} - \lambda)\varphi_m(\lambda) = 0$ such that $\varphi_m(\lambda, m) = 0$ and $\varphi_m(\lambda, n) \to u_{1,-}(\lambda, n)$ holds at $n = -N-1, \ldots, N+1$.
By Lemma 8.26, $W_{-N}(u_{0,+}(\lambda), u_{1,-}(\lambda)) \neq 0$, and $W_N(u_{0,+}(\lambda), u_{1,-}(\lambda)) \neq 0$ we have $\#_{(-N,N]}(u_{0,+}(\lambda), \varphi_m(\lambda)) = \#_{(-N,N]}(u_{0,+}(\lambda), u_{1,-}(\lambda))$ for all m sufficiently large and thus

$$\mathcal{C}_{0,z}(\lambda) = \#_{(m,\infty]}(\psi_{0,m,+}(\lambda), \varphi_m(\lambda)) = \#_{(-N,N]}(u_{0,+}(\lambda), \varphi_m(\lambda)) = \mathcal{N}(\lambda)$$

since $W(\psi_{0,m,+}(\lambda), \varphi_m(\lambda))$ is constant and nonzero for all $j \leqslant -N$, $j \geqslant N$. □

Lemma 11.8. *Let* $[z_-, z_+] \cap \sigma_{ess}(H_0) = \emptyset$ *and* $\dim \operatorname{Ran}(H_0 - H_1) < \infty$, *then*

$$E_{[z_-,z_+)}(H_1) - E_{(z_-,z_+]}(H_0) = \mathcal{N}(z_+) - \mathcal{N}(z_-).$$

Proof. Let $\lambda_- < \lambda_+, \lambda_\pm \in (z_-, z_+)$ such that $\lambda_\pm \in \rho(H_0) \cap \rho(H_1)$. Then, by Lemma 11.7 and Lemma 11.1 we have

$$\begin{aligned} \mathcal{C}_{0,z_+}(\lambda_+) &= \mathcal{N}(\lambda_+) = \mathcal{N}(z_+) - E_{(\lambda_+,z_+)}(H_1) + E_{(\lambda_+,z_+]}(H_0), \\ \mathcal{C}_{0,z_-}(\lambda_-) &= \mathcal{N}(\lambda_-) = \mathcal{N}(z_-) + E_{[z_-,\lambda_-)}(H_1) - E_{(z_-,\lambda_-)}(H_0). \end{aligned}$$

Now, by Corollary 11.5 we have

$$\begin{aligned} &E_{[z_-,z_+)}(H_1) - E_{(z_-,z_+]}(H_0) \\ &\quad = E_{[z_-,\lambda_-)}(H_1) - E_{(z_-,\lambda_-]}(H_0) + E_{[\lambda_-,\lambda_+)}(H_1) \\ &\qquad - E_{(\lambda_-,\lambda_+]}(H_0) + E_{[\lambda_+,z_+)}(H_1) - E_{(\lambda_+,z_+]}(H_0) \\ &\quad = \mathcal{N}(\lambda_-) - \mathcal{N}(z_-) + \mathcal{N}(\lambda_+) - \mathcal{N}(\lambda_-) + \mathcal{N}(z_+) - \mathcal{N}(\lambda_+) \\ &\quad = \mathcal{N}(z_+) - \mathcal{N}(z_-). \end{aligned}$$

□

11.3 Main theorem for infinite operators

Now we look at a rank-one perturbation of an infinite Jacobi operator, and give a self-contained proof, which shows, how an eigenvalue is moved to a different position.

Lemma 11.9. *Let $z_- < z < z_+$, $z \in \sigma_d(H)$, $[z_-, z_+] \cap \sigma(H) = \{z\}$, $\varepsilon \neq 0$, and*

$$H_\varepsilon = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & b(0) & a(0) & & \\ & a(0) & b(1)+\varepsilon & a(1) & \\ & & a(1) & b(2) & \ddots \\ & & & \ddots & \ddots \end{pmatrix}. \tag{11.21}$$

Then,

$$z \in \sigma(H_\varepsilon) \iff u_+(z,1) = 0 \iff u_-(z,1) = 0 \tag{11.22}$$

$$\iff z \in \sigma(H_{1,+}) \iff z \in \sigma(H_{-,1}). \tag{11.23}$$

If $W_0(u_+(z_\pm), u_-(z_\pm))$ and $W_0(u_+(z_\pm), u_-(z_\pm)) - \varepsilon u_+(z_\pm, 1) u_-(z_\pm, 1)$ are of the same sign (and non-zero), then

$$\begin{aligned} E_{[z_-,z_+]}(H_\varepsilon) &= E_{(z_-,z_+)}(H_\varepsilon) = 1 \\ &= \begin{cases} E_{(z,z_+)}(H_\varepsilon) & \text{if } \varepsilon > 0,\ u_+(z,1) \neq 0 \\ E_{(z_-,z)}(H_\varepsilon) & \text{if } \varepsilon < 0,\ u_+(z,1) \neq 0 \\ E_{\{z\}}(H_\varepsilon) & \text{otherwise.} \end{cases} \end{aligned} \tag{11.24}$$

Proof. The Wronskian $W(u_+(z), u_-(z))$ vanishes and $W(u_+(z_\pm), u_-(z_\pm))$ is constant and nonvanishing, thus we have $\#_{(-\infty,\infty]}(u_+(z), u_-(z)) = -1$ and $\#_{(-\infty,\infty]}(u_+(z_\pm), u_-(z_\pm)) = 0$. For all $\lambda \notin \sigma_{ess}(H)$ there exists a solution $u_{\varepsilon,\pm}(\lambda)$ of $(\tau_\varepsilon - \lambda)u = 0$ such that $u_{\varepsilon,\pm}(\lambda, j) = u_\pm(\lambda, j)$ for all $\pm j \geqslant \pm 1$. Hence, $W_j(u_{\varepsilon,+}(\lambda), u_-(\lambda)) = W_j(u_+(\lambda), u_-(\lambda))$ for all $j \geqslant 1$. And $W_j(u_{\varepsilon,+}(\lambda), u_-(\lambda))$ is constant for all $j \leqslant 0$ by

$$\begin{aligned} &W_j(u_{\varepsilon,+}(\lambda), u_-(\lambda)) - W_{j-1}(u_{\varepsilon,+}(\lambda), u_-(\lambda)) \\ &\quad = (b(j) - b(j)) u_{\varepsilon,+}(\lambda, j) u_-(\lambda, j) = 0 \end{aligned}$$

and we have

$$\begin{aligned} &W_0(u_+(\lambda), u_-(\lambda)) - W_0(u_{\varepsilon,+}(\lambda), u_-(\lambda)) \\ &\quad = W_1(u_{\varepsilon,+}(\lambda), u_-(\lambda)) - W_0(u_{\varepsilon,+}(\lambda), u_-(\lambda)) = \varepsilon u_{\varepsilon,+}(\lambda, 1) u_-(\lambda, 1). \end{aligned}$$

If either $W_0(u_+(z_\pm), u_-(z_\pm))$ and $W_0(u_{\varepsilon,+}(z_\pm), u_-(z_\pm))$ both are positive or both are negative, then

$$\#_{(-\infty,\infty]}(u_{\varepsilon,+}(z_\pm), u_-(z_\pm)) = \#_{(-\infty,\infty]}(u_+(z_\pm), u_-(z_\pm)) = 0$$

holds and hence by Lemma 11.8 we have

$$\begin{aligned} &1 - E_{(z_-,z_+]}(H_\varepsilon) = E_{[z_-,z_+)}(H) - E_{(z_-,z_+]}(H_\varepsilon) \\ &\quad = \#_{(-\infty,\infty]}(u_{\varepsilon,+}(z_+), u_-(z_+)) - \#_{(-\infty,\infty]}(u_{\varepsilon,+}(z_-), u_-(z_-)) = 0. \end{aligned}$$

By $W(u_{\varepsilon,+}(z_\pm), u_-(z_\pm)) = W(u_{\varepsilon,+}(z_\pm), u_{\varepsilon,-}(z_\pm))$ is nonvanishing near $-\infty$ we have $z_\pm \in \rho(H_\varepsilon)$, thus $E_{[z_-,z_+]}(H_\varepsilon) = E_{(z_-,z_+)}(H_\varepsilon) = 1$. Now, by $\varepsilon \neq 0$ and $W_0(u_{\varepsilon,+}(z), u_-(z)) = -\varepsilon u_+(z,1)^2$ we have

$$\#_{(-\infty,\infty]}(u_{\varepsilon,+}(z), u_-(z)) = \begin{cases} -1 & \text{if } \varepsilon < 0 \text{ or } u_+(z,1) = 0, \\ 0 & \text{if } \varepsilon > 0. \end{cases}$$

Moreover, by $u_-(z,j) = u_{\varepsilon,-}(z,j)$ for all $j \leqslant 1$ we have

$$\begin{aligned} z \in \sigma(H_\varepsilon) \iff & W(u_{\varepsilon,+}(z), u_{\varepsilon,-}(z)) \equiv 0 = W_0(u_{\varepsilon,+}(z), u_-(z)) \\ \iff & u_+(z,1) = 0 \iff u_-(z,1) = 0 \\ \iff & z \in \sigma(H_{1,+}) \iff z \in \sigma(H_{-,1}). \end{aligned} \tag{11.25}$$

If $u_+(z,1) \neq 0$, then by Lemma 11.8 we now have

$$\begin{aligned} &0 - E_{(z_-,z]}(H_\varepsilon) = E_{[z_-,z)}(H) - E_{(z_-,z]}(H_\varepsilon) \\ &\quad = \#_{(-\infty,\infty]}(u_{\varepsilon,+}(z), u_-(z)) - \#_{(-\infty,\infty]}(u_{\varepsilon,+}(z_-), u_-(z_-)) \\ &\quad = - \begin{cases} 1 & \text{if } \varepsilon < 0 \\ 0 & \text{if } \varepsilon > 0. \end{cases} \end{aligned}$$

Hence, $E_{(z_-,z)}(H_\varepsilon) = E_{(z_-,z]}(H_\varepsilon) = 1$ if $\varepsilon < 0$ and $E_{(z_-,z]}(H_\varepsilon) = 0$ if $\varepsilon > 0$. □

The criterion on the signs of the Wronskian from the previous lemma can also be formulated in terms of the Green function, see

Remark 11.10. *The Wonskians $W(u_+(\lambda), u_-(\lambda))$ and $W(u_{\varepsilon,+}(\lambda), u_{\varepsilon,-}(\lambda))$ are* *constant and we have*

$$W(u_{\varepsilon,+}(\lambda), u_{\varepsilon,-}(\lambda)) = W(u_+(\lambda), u_-(\lambda)) - \varepsilon u_+(\lambda, 1)u_-(\lambda, 1). \tag{11.26}$$

If $\lambda \in \rho(H)$, then $W(u_+(\lambda), u_-(\lambda)) \neq 0$ and

$$G_H(\lambda, 1, 1) = \frac{u_+(\lambda, 1)u_-(\lambda, 1)}{W(u_-(\lambda), u_+(\lambda))}$$

exists. If so, then by

$$0 < \frac{W(u_+(\lambda), u_-(\lambda)) - \varepsilon u_+(\lambda, 1)u_-(z_\pm, 1)}{W(u_+(\lambda), u_-(\lambda))} = 1 + \varepsilon \frac{u_+(\lambda, 1)u_-(\lambda, 1)}{W(u_-(\lambda), u_+(\lambda))}$$

both Wronskians are of the same sign (and non-zero) if and only if

$$-1 < \varepsilon G_H(\lambda, 1, 1). \tag{11.27}$$

If an infinite Jacobi operator H has an eigenvalue at z, then, in the approximating sequence there's a semi-infinite Jacobi operator of sufficiently large dimension, which has an eigenvalue near z, since the semi-infinite operators converge in strong resolvent sense.

Lemma 11.11. *Let $z_- < z < z_+$, $[z_-, z_+] \cap \sigma_{ess}(H) = \emptyset$, and $z \in \sigma(H)$. Then, for all $N \in \mathbb{Z}$, there exists an $M < N$, $\tilde{z} \in (z_-, z_+)$ such that $\tilde{z} \neq z$ and*

$$u_+(z, M) \neq 0, \quad u_+(\tilde{z}, M) = 0.$$

Proof. Let $\mathcal{J} = \{n \in \mathbb{Z} \mid u_+(z, n) \neq 0\}$, then $\mathcal{J}$ is an infinite set. Let $z_0 \notin [z_-, z_+]$, then $z_0\mathbb{I} \oplus H_{n,+} \stackrel{sr}{\to} H$ as $n \to -\infty, n \in \mathcal{J}$, by Theorem 2.21.b. Thus, by Theorem 2.9 and Lemma 2.19 we have

$$\liminf_{\substack{n\to-\infty \\ n\in\mathcal{J}}} E_{(z_-,z_+)}(H_{n,+}) = \liminf_{\substack{n\to-\infty \\ n\in\mathcal{J}}} E_{(z_-,z_+)}(z_0\mathbb{I} \oplus H_{n,+}) \geqslant E_{(z_-,z_+)}(H) \geqslant 1.$$

Hence, for all N there exists an $M < N, M \in \mathcal{J}$, such that $E_{(z_-,z_+)}(H_{M,+}) \geqslant 1$. By $u_+(z, M) \neq 0$ we have $z \notin \sigma(H_{M,+})$, thus there exists some $\tilde{z} \neq z$, $\tilde{z} \in (z_-, z_+) \cap \sigma(H_{M,+})$. Now, $u_+(\tilde{z}, M) = 0$ holds. □

Now, we're ready to show, that the assumption (1.21) can be dropped if we look at the vicinity of a point, which is in the spectra of both Jacobi operators.

Lemma 11.12. *Let $b_0 \downarrow b_1$ near $+\infty$ and near $-\infty$, $z_- < z < z_+$, $z \in \sigma_d(H_j)$, and $[z_-, z_+] \cap \sigma(H_j) = \{z\}$, where $j = 0, 1$, then we have*

$$E_{[z_-,z)}(H_1) - E_{(z_-,z]}(H_0) = \mathcal{N}(z) - \mathcal{N}(z_-),$$
$$E_{[z,z_+)}(H_1) - E_{(z,z_+]}(H_0) = \mathcal{N}(z_+) - \mathcal{N}(z).$$

Proof. Let M, $\tilde{z} \in (z_-, z_+)$, $\tilde{z} \neq z$, so that

$$u_{0,+}(z, M) \neq 0 \quad \text{and} \quad u_{0,+}(\tilde{z}, M) = 0$$

and moreover $b_0 - b_1 \geqslant 0$ and $W(u_{0,+}(z), u_{1,-}(z))$ is of one sign (or vanishing) to the left of M (use Lemma 11.11). Let $\varepsilon < 0$ so that

$$-\frac{1}{\varepsilon} > G_H(\lambda, M, M) \quad \text{at } \lambda = z_\pm, \tilde{z} \tag{11.28}$$

and, if $W_{M-1}(u_{0,+}(z), u_{1,-}(z)) \neq 0$, such that $W_{M-1}(u_{0,+}(z), u_{1,-}(z))$ and $W_{M-1}(u_{0,+}(z), u_{1,-}(z)) - \varepsilon u_{0,+}(z, M) u_{1,-}(z, M)$ are of the same sign. If $(b_0 - b_1)(M) \neq 0$, then we further assume $-\varepsilon < \frac{(b_0 - b_1)(M)}{2}$. Let

$$\tilde{H}_0 = \begin{pmatrix} \ddots & & \ddots & & & \\ \ddots & b_0(M-1) & a(M-1) & & & \\ & a(M-1) & b_0(M) + \varepsilon & a(M) & & \\ & & a(M) & b_0(M+1) & \ddots & \\ & & & \ddots & \ddots & \end{pmatrix}, \tag{11.29}$$

then the solutions $\tilde{u}_+$ and u_+ coincide above M and moreover by $u_{0,+}(\tilde{z}, M) = 0$ the solutions at $\tilde{z}$ actually coincide everywhere. Hence, at $\tilde{z}$ also the Wronskians coincide everywhere and by comparing the weights of their nodes (there is no node at $M - 1$ by $W_{M-1}(u_{0,+}(\tilde{z}), u_{1,-}(\tilde{z})) = W_M(u_{0,+}(\tilde{z}), u_{1,-}(\tilde{z})))$ we find

$$\#_{(-\infty,\infty]}(\tilde{u}_{0,+}(\tilde{z}), u_{1,-}(\tilde{z})) = \mathcal{N}(\tilde{z}). \tag{11.30}$$

By Lemma 11.6 the Wronskian $W(\tilde{u}_{0,+}(z), u_{1,-}(z))$ is not vanishing near $-\infty$ and by $b_0 - b_1 \geqslant 0$ to the left of M we have

$$\begin{aligned} &\#_{(-\infty,\infty]}(\tilde{u}_{0,+}(z), u_{1,-}(z)) \geqslant \#_{[M-1,\infty]}(\tilde{u}_{0,+}(z), u_{1,-}(z)), \\ &\#_{[M,\infty]}(\tilde{u}_{0,+}(z), u_{1,-}(z)) = \#_{[M,\infty]}(u_{0,+}(z), u_{1,-}(z)) \geqslant \mathcal{N}(z), \end{aligned}$$

hence it remains to look at a possible node at $M - 1$: if $(b_0 - b_1)(M) > 0$, then $(b_0 + \varepsilon - b_1)(M) > 0$ and hence $\#_{M-1}(\tilde{u}_{0,+}(z), u_{1,-}(z)) \geqslant 0$. If $(b_0 - b_1)(M) = 0$ and $W_M(u_{0,+}(z), u_{1,-}(z)) \neq 0$, then $W_M(\tilde{u}_{0,+}(z), u_{1,-}(z))$ and $W_{M-1}(\tilde{u}_{0,+}(z), u_{1,-}(z))$ are of the same sign, thus $\#_{M-1}(\tilde{u}_{0,+}(z), u_{1,-}(z)) = 0$. If $(b_0 - b_1)(M) = W_M(u_{0,+}(z), u_{1,-}(z)) = 0$, then $W(u_{0,+}(z), u_{1,-}(z))$ vanishes near $-\infty$, thus $\#_{(-\infty,\infty]}(\tilde{u}_{0,+}(z), u_{1,-}(z)) = \#_{[M,\infty]}(u_{0,+}(z), u_{1,-}(z)) - 1 = \mathcal{N}(z)$. Hence, in either case we have

$$\#_{(-\infty,\infty]}(\tilde{u}_{0,+}(z), u_{1,-}(z)) \geqslant \mathcal{N}(z).$$

By Remark 11.10 and Lemma 11.9 we have

$$E_{[z_-,z_+]}(\tilde{H}_0) = 1 = \begin{cases} E_{(\tilde{z},z)}(\tilde{H}_0) & \text{if } \tilde{z} < z \\ E_{(z_-,z)}(\tilde{H}_0) & \text{if } z < \tilde{z}. \end{cases}$$

Thus, by Lemma 11.2 and Corollary 11.5 if $\tilde{z} < z$, then

$$\begin{aligned} \mathcal{N}(z) - \mathcal{N}(\tilde{z}) &\geqslant E_{(\tilde{z},z)}(H_1) - E_{(\tilde{z},z]}(H_0) \\ &= E_{[\tilde{z},z)}(H_1) - E_{(\tilde{z},z]}(H_0) = E_{[\tilde{z},z)}(H_1) - E_{(\tilde{z},z]}(\tilde{H}_0) \\ &= \#_{(-\infty,\infty]}(\tilde{u}_{0,+}(z), u_{1,-}(z)) - \#_{(-\infty,\infty]}(\tilde{u}_{0,+}(\tilde{z}), u_{1,-}(\tilde{z})) \geqslant \mathcal{N}(z) - \mathcal{N}(\tilde{z}) \end{aligned} \tag{11.31}$$

and if $z < \tilde{z}$, then

$$\begin{aligned} \mathcal{N}(\tilde{z}) - \mathcal{N}(z) &\leqslant E_{[z,\tilde{z})}(H_1) - E_{(z,\tilde{z})}(H_0) \\ &= E_{[z,\tilde{z})}(H_1) - E_{(z,\tilde{z}]}(H_0) = E_{[z,\tilde{z})}(H_1) - E_{(z,\tilde{z}]}(\tilde{H}_0) \\ &= E_{[z_-,\tilde{z})}(H_1) - E_{(z_-,\tilde{z}]}(\tilde{H}_0) - (E_{[z_-,z)}(H_1) - E_{(z_-,z]}(\tilde{H}_0)) \\ &= \#_{(-\infty,\infty]}(\tilde{u}_{0,+}(\tilde{z}), u_{1,-}(\tilde{z})) - \#_{(-\infty,\infty]}(\tilde{u}_{0,+}(z_-), u_{1,-}(z_-)) \\ &\quad - (\#_{(-\infty,\infty]}(\tilde{u}_{0,+}(z), u_{1,-}(z)) - \#_{(-\infty,\infty]}(\tilde{u}_{0,+}(z_-), u_{1,-}(z_-))) \\ &\leqslant \mathcal{N}(\tilde{z}) - \mathcal{N}(z). \end{aligned} \tag{11.32}$$

In the first case we now obtain our claim by

$$\begin{aligned} &E_{[z_-,z)}(H_1) - E_{(z_-,z]}(H_0) \\ &\quad = E_{[z_-,\tilde{z})}(H_1) - E_{(z_-,\tilde{z}]}(H_0) + E_{[\tilde{z},z)}(H_1) - E_{(\tilde{z},z]}(H_0) = \mathcal{N}(z) - \mathcal{N}(z_-), \\ &E_{[z,z_+)}(H_1) - E_{(z,z_+]}(H_0) \\ &\quad = E_{[\tilde{z},z_+)}(H_1) - E_{(\tilde{z},z_+]}(H_0) - (E_{[\tilde{z},z)}(H_1) - E_{(\tilde{z},z]}(H_0)) = \mathcal{N}(z_+) - \mathcal{N}(z), \end{aligned}$$

and in the second case by

$$\begin{aligned} &E_{[z,z_+)}(H_1) - E_{(z,z_+]}(H_0) \\ &\quad = E_{[z,\tilde{z})}(H_1) - E_{(z,\tilde{z}]}(H_0) + E_{[\tilde{z},z_+)}(H_1) - E_{(\tilde{z},z_+]}(H_0) = \mathcal{N}(z_+) - \mathcal{N}(z), \\ &E_{[z_-,z)}(H_1) - E_{(z_-,z]}(H_0) \\ &\quad = E_{[z_-,\tilde{z})}(H_1) - E_{(z_-,\tilde{z}]}(H_0) - (E_{[z,\tilde{z})}(H_1) - E_{(z,\tilde{z}]}(H_0)) = \mathcal{N}(z) - \mathcal{N}(z_-). \end{aligned}$$

□

Finally, the following lemma completes the proof of Theorem 1.1.

Lemma 11.13. *Let $[z_-, z_+] \cap \sigma_{ess}(H_0) = \emptyset$ and let $b_0 \downarrow b_1$ near $+\infty$ and near $-\infty$, then*

$$E_{[z_-,z_+)}(H_1) - E_{(z_-,z_+]}(H_0) = \mathcal{N}(z_+) - \mathcal{N}(z_-). \tag{11.33}$$

Proof of Lemma 11.13, which is (1.13). Let $\varepsilon_+ > 0$ be sufficiently small such that

$$[z_+ - \varepsilon_+, z_+ + \varepsilon_+] \cap (\sigma(H_0) \cup \sigma(H_1)) \subseteq \{z_+\}$$

and let $\alpha = z_+ - \varepsilon_+, \beta = z_+ + \varepsilon_+$. If $z_+ \in \sigma(H_0) \cap \sigma(H_1)$ or $z_+ \notin \sigma(H_0)$, then by Lemma 11.12 and Lemma 11.2 we have $E_{[\alpha,z_+)}(H_1) - E_{(\alpha,z_+]}(H_0) = \mathcal{N}(z_+) - \mathcal{N}(\alpha)$. If $z_+ \notin \sigma(H_1)$, then by Lemma 11.2 $E_{[z_+,\beta)}(H_1) - E_{(z_+,\beta]}(H_0) = \mathcal{N}(\beta) - \mathcal{N}(z_+)$ holds and hence by $E_{[\alpha,\beta)}(H_1) - E_{(\alpha,\beta]}(H_0) = \mathcal{N}(\beta) - \mathcal{N}(\alpha)$ we have

$$\begin{aligned} &E_{[\alpha,z_+)}(H_1) - E_{(\alpha,z_+]}(H_0) \\ &\quad = E_{[\alpha,\beta)}(H_1) - E_{(\alpha,\beta]}(H_0) - (E_{[z_+,\beta)}(H_1) - E_{(z_+,\beta]}(H_0)) \\ &\quad = \mathcal{N}(\beta) - \mathcal{N}(\alpha) - (\mathcal{N}(\beta) - \mathcal{N}(z_+)) = \mathcal{N}(z_+) - \mathcal{N}(\alpha). \end{aligned} \tag{11.34}$$

Let $\varepsilon_- > 0$ be sufficiently small such that

$$[z_- - \varepsilon_-, z_- + \varepsilon_-] \cap (\sigma(H_0) \cup \sigma(H_1)) \subseteq \{z_-\}$$

and let $\gamma = z_- - \varepsilon_-, \delta = z_- + \varepsilon_-$. If $z_- \in \sigma(H_0) \cap \sigma(H_1)$ or $z_- \notin \sigma(H_1)$, then by Lemma 11.12 and Lemma 11.2 we have $E_{[z_-,\delta)}(H_1) - E_{(z_-,\delta]}(H_0) = \mathcal{N}(\delta) - \mathcal{N}(z_-)$. If $z_- \notin \sigma(H_0)$, then by Lemma 11.2 $E_{[\gamma,z_-)}(H_1) - E_{(\gamma,z_-]}(H_0) = \mathcal{N}(z_-) - \mathcal{N}(\gamma)$ holds and hence by $E_{[\gamma,\delta)}(H_1) - E_{(\gamma,\delta]}(H_0) = \mathcal{N}(\delta) - \mathcal{N}(\gamma)$ we have

$$\begin{aligned} &E_{[z_-,\delta)}(H_1) - E_{(z_-\delta]}(H_0) \\ &\quad = E_{[\gamma,\delta)}(H_1) - E_{(\gamma,\delta]}(H_0) - (E_{[\gamma,z_-)}(H_1) - E_{(\gamma,z_-]}(H_0)) \\ &\quad = \mathcal{N}(\delta) - \mathcal{N}(\gamma) - (\mathcal{N}(z_-) - \mathcal{N}(\gamma)) = \mathcal{N}(\delta) - \mathcal{N}(z_-). \end{aligned} \tag{11.35}$$

By Lemma 11.2 we have $E_{[\delta,\alpha)}(H_1) - E_{(\delta,\alpha]}(H_0) = \mathcal{N}(\alpha) - \mathcal{N}(\delta)$ and thus,

$$\begin{aligned} &E_{[z_-,z_+)}(H_1) - E_{(z_-,z_+]}(H_0) \\ &\quad = E_{[z_-,\delta)}(H_1) - E_{(z_-\delta]}(H_0) + E_{[\delta,\alpha)}(H_1) - E_{(\delta,\alpha]}(H_0) \\ &\qquad + E_{[\alpha,z_+)}(H_1) - E_{(\alpha,z_+]}(H_0) \\ &\quad = \mathcal{N}(\delta) - \mathcal{N}(z_-) + \mathcal{N}(\alpha) - \mathcal{N}(\delta) + \mathcal{N}(z_+) - \mathcal{N}(\alpha) = \mathcal{N}(z_+) - \mathcal{N}(z_-). \end{aligned}$$

□

Appendix A

Linear interpolation

Let $\tau_\varepsilon, \varepsilon \in [0,1]$, denote the difference equations which arise from linear interpolation of the coefficients a_0, b_0 and a_1, b_1, that is,

$$a_\varepsilon = a_0 - \varepsilon\underbrace{(a_0 - a_1)}_{=\Delta a} \quad \text{and} \quad b_\varepsilon = b_0 - \varepsilon\underbrace{(b_0 - b_1)}_{=\Delta b}. \tag{A.1}$$

Clearly, $a_0, a_1 < 0$ implies $a_\varepsilon < 0$ and hence τ_ε corresponds to a Jacobi matrix

$$H^\varepsilon_{n_0,n} = H^0_{n_0,n} - \varepsilon\underbrace{(H^0_{n_0,n} - H^1_{n_0,n})}_{=\Delta H_{n_0,n}}, \tag{A.2}$$

where $H_{n_0,n}$ are the matrices from (2.43). The perturbation matrix $\Delta H_{n_0,n}$ is tridiagonal and symmetric, but not necessarily a Jacobi matrix (i.e. some elements of a could be zero). Now, fix initial values $u(n_0), u(n_0+1) \in \mathbb{R}$ and let u_ε be the solution of $(\tau_\varepsilon - z)u_\varepsilon = 0$ fulfilling

$$u_\varepsilon(n_0) = u(n_0), \quad u_\varepsilon(n_0+1) = u(n_0+1). \tag{A.3}$$

Lemma A.1. *Let $n \in \mathbb{Z}$ and*

$$\begin{aligned} \Xi_n : [0,1] &\to \mathbb{R} \\ \varepsilon &\mapsto u_\varepsilon(n), \end{aligned} \tag{A.4}$$

then

$$\Xi_n \in C^1([0,1], \mathbb{R}). \tag{A.5}$$

Proof. We use mathematical induction: the claim holds at $n = n_0$ and $n = n_0+1$ since Ξ_{n_0} and Ξ_{n_0+1} are constant. For all $n > n_0+1$, respectively $n < n_0$, by

$(\tau_\varepsilon - z)u_\varepsilon = 0$ we have

$$u_\varepsilon(n) = \frac{1}{a_\varepsilon(n-1)}(-a_\varepsilon(n-2)u_\varepsilon(n-2) - (b_\varepsilon(n-1)-z)u_\varepsilon(n-1)) \tag{A.6}$$

and $u_\varepsilon(n) = \frac{1}{a_\varepsilon(n)}(-a_\varepsilon(n+1)u_\varepsilon(n+2) - (b_\varepsilon(n+1)-z)u_\varepsilon(n+1))$. Assume the claim holds at $n_0, \ldots, n-1$, respectively at $n+1, \ldots, n_0$, then we have $\Xi_n \in C^1([0,1], \mathbb{R})$ for all $n \in \mathbb{Z}$ by $a_\varepsilon < 0$. □

Let the dot denote the derivative of $u_\varepsilon(n)$ with respect to ε, that is,

$$\dot{u}_\varepsilon(n) = \lim_{r \to \varepsilon} \frac{u_r(n) - u_\varepsilon(n)}{r - \varepsilon}. \tag{A.7}$$

Lemma A.2. *There exist unique sequences $\rho_\varepsilon, \theta_\varepsilon \in \ell(\mathbb{Z}, \mathbb{R})$ where*

$$\rho_\varepsilon(n), \theta_\varepsilon(n) \in C^1([0,1], \mathbb{R}) \tag{A.8}$$

for all $n \in \mathbb{Z}$. Moreover,

$$\begin{aligned} u_\varepsilon(n) &= \rho_\varepsilon(n) \sin\theta_\varepsilon(n), \\ -a_\varepsilon(n)u_\varepsilon(n+1) &= \rho_\varepsilon(n)\cos\theta_\varepsilon(n), \end{aligned} \tag{A.9}$$

where $\rho_\varepsilon > 0$, $\theta_\varepsilon(n_0) \in (-\pi, \pi]$ is constant, and

$$\lceil \theta_\varepsilon(n)/\pi \rceil \leqslant \lceil \theta_\varepsilon(n+1)/\pi \rceil \leqslant \lceil \theta_\varepsilon(n)/\pi \rceil + 1 \tag{A.10}$$

holds for all $n \in \mathbb{Z}$.

Proof. At $\varepsilon = 0$ let ρ_0, θ_0 be the Prüfer variables of u_0 as introduced in (3.19). By the previous lemma the function

$$\begin{aligned} f_n &: [0,1] \to \mathbb{R}^2 \\ &\varepsilon \mapsto f_n(\varepsilon) = (-a_\varepsilon(n)u_\varepsilon(n+1), u_\varepsilon(n)) \neq (0,0), \end{aligned} \tag{A.11}$$

is continuously differentiable with respect to ε in each component. Let $\rho_\varepsilon(n)$ and $\theta_\varepsilon(n)$ be the polar coordinates of $f_n(\varepsilon)$ such that

$$\theta_\varepsilon(n) = \arg f_n(\varepsilon) + k_n(\varepsilon)2\pi \tag{A.12}$$

where $\arg f_n(\varepsilon) \in (-\pi, \pi]$ is the principal value and $k_n(\varepsilon) \in \mathbb{Z}$ is choosen such that $\rho_\varepsilon(n)$ and $\theta_\varepsilon(n)$ are continuous with respect to ε. Then, $\rho_\varepsilon(n)$ and $\theta_\varepsilon(n)$ are continuously differentiable since $f_n(\varepsilon)$ is, $\theta_\varepsilon(n_0) \in (-\pi, \pi]$ is constant, and (A.9) holds. It remains to show that for all n either

$$\lceil \theta_\varepsilon(n+1)/\pi \rceil = \lceil \theta_\varepsilon(n)/\pi \rceil \quad \text{or} \quad \lceil \theta_\varepsilon(n+1)/\pi \rceil = \lceil \theta_\varepsilon(n)/\pi \rceil + 1 \tag{A.13}$$

holds: therefore fix some $n \in \mathbb{Z}$. First of all, note, that if the claim holds at some $\varepsilon_0 \in [0,1]$, then by Lemma 3.10 there exists some $k \in \mathbb{Z}$ such that

$$\begin{aligned} \theta_{\varepsilon_0}(n) \in k\pi + (0, \frac{\pi}{2}], \quad \theta_{\varepsilon_0}(n+1) \in k\pi + (0, \pi] \\ \iff u_{\varepsilon_0} \text{ has no node at } n, \\ \theta_{\varepsilon_0}(n) \in k\pi + (\frac{\pi}{2}, \pi], \quad \theta_{\varepsilon_0}(n+1) \in k\pi + (\pi, 2\pi) \\ \iff u_{\varepsilon_0} \text{ has a node at } n. \end{aligned} \tag{A.14}$$

Now, let X_n (resp. X_{n+1}) be the (by continuity) closed subset of $[0,1]$ where $u_\varepsilon(n)$ (resp. $u_\varepsilon(n+1)$) vanishes. Since the zeros of u are simple we have

$$X_n \cap X_{n+1} = \emptyset.$$

Let O be a connected component of $[0,1] \setminus (X_n \cup X_{n+1})$ and suppose (A.13) holds at some $\varepsilon \in O$. Then, by continuity of $\theta_\varepsilon(n) \neq 0 \mod \pi$ and $\theta_\varepsilon(n+1) \neq 0 \mod \pi$ (A.13) holds for all $\varepsilon \in O$.

Let C_n be a connected component of X_n and suppose that in every vicinity of C_n there exists an ε_0 such that (A.13) holds at ε_0: since $\theta_\varepsilon(n)$ is continuous and X_n, X_{n+1} are closed disjoint sets there exists a vicinity V of C_n such that

$$\theta_\varepsilon(n) \in (l\pi - \frac{\pi}{2}, l\pi + \frac{\pi}{2}), \quad l \in \mathbb{Z},$$

and $\theta_\varepsilon(n+1) \neq 0 \mod \pi$ holds for all $\varepsilon \in V$. Now, choose $\varepsilon_0 \in V$ such that (A.13) holds at ε_0, then (A.14) holds at ε_0 and hence we have $\theta_{\varepsilon_0}(n+1) \in (l\pi, (l+1)\pi)$. Since $\theta_\varepsilon(n+1)$ is continuous and $\theta_\varepsilon(n+1) \neq 0 \mod \pi$ we have $\theta_\varepsilon(n+1) = (l\pi, (l+1)\pi)$ for all $\varepsilon \in V$. Hence, there exists a vicinity V of C_n such that (A.13) holds for all $\varepsilon \in V$.

Let C_{n+1} be a connected component of X_{n+1} and suppose that in every vicinity of C_{n+1} there exists an ε_0 such that (A.13) holds at ε_0: since $\theta_\varepsilon(n+1)$ is continuous and X_n, X_{n+1} are closed disjoint sets there exists a vicinity V of C_{n+1} such that

$$\theta_\varepsilon(n+1) \in (l\pi - \frac{\pi}{2}, l\pi + \frac{\pi}{2}), \quad l \in \mathbb{Z},$$

and $u_\varepsilon(n)u_\varepsilon(n+2) < 0$ (and hence $\sin\theta_\varepsilon(n)\cos\theta_\varepsilon(n+1) < 0$) holds for all $\varepsilon \in V$. Now, choose $\varepsilon_0 \in V$ such that (A.13) holds at ε_0 and hence by (A.14) and $\sin\theta_\varepsilon(n)\cos\theta_\varepsilon(n+1) < 0$ we have $\theta_{\varepsilon_0}(n) \in ((l-1)\pi, l\pi)$. Since $\theta_\varepsilon(n)$ is continuous and $\theta_\varepsilon(n) \neq 0 \mod \pi$ we have $\theta_\varepsilon(n) = ((l-1)\pi, l\pi)$ for all $\varepsilon \in V$. Hence, there exists a vicinity V of C_{n+1} such that (A.13) holds for all $\varepsilon \in V$.

Since the union of the mentioned vicinitys V of all connected components of X_n and X_{n+1} and the open set $[0,1] \setminus (X_n \cup X_{n+1})$ is a cover of $[0,1]$ the claim (A.13) now holds for all $\varepsilon \in [0,1]$ since it holds at $\varepsilon = 0$. □

A.1 Derivative of the Prüfer angle

Consider the solutions s_ε of $(\tau_\varepsilon - z)s_\varepsilon = 0$ with initial values $s_\varepsilon(n_0) = 0$ and $s_\varepsilon(n_0+1) = 1$.

Lemma A.3. *Let $n > n_0 + 2$ and $\vec{s}_\varepsilon = s_\varepsilon\big|_{\ell(n_0,n+1)}$, then*

$$\begin{aligned} a_\varepsilon(n)s_\varepsilon(n)\dot{s}_\varepsilon(n+1) - \frac{d}{d\varepsilon}(a_\varepsilon(n)s_\varepsilon(n))s_\varepsilon(n+1) \\ = \langle \vec{s}_\varepsilon, \Delta H_{n_0,n+1}\vec{s}_\varepsilon\rangle + 2\Delta a(n)s_\varepsilon(n)s_\varepsilon(n+1). \end{aligned} \tag{A.15}$$

Proof. We have

$$\begin{aligned} & a_\varepsilon(n)s_\varepsilon(n)\dot{s}_\varepsilon(n+1) - \frac{d}{d\varepsilon}(a_\varepsilon(n)s_\varepsilon(n))s_\varepsilon(n+1) \\ &= \lim_{r\to\varepsilon} \frac{a_\varepsilon(n)s_\varepsilon(n)s_r(n+1) - a_r(n)s_r(n)s_\varepsilon(n+1)}{r-\varepsilon} \\ &= \lim_{r\to\varepsilon} M_n^{\varepsilon,r}(s_\varepsilon,s_r)(r-\varepsilon)^{-1} \\ &= \lim_{r\to\varepsilon}(r-\varepsilon)^{-1}(W_{n+1}^{\varepsilon,r}(s_\varepsilon,s_r) - W_0^{\varepsilon,r}(s_\varepsilon,s_r) \\ &\quad - (b_\varepsilon(n+1) - b_r(n+1))s_\varepsilon(n+1)s_r(n+1)) \\ &= \lim_{r\to\varepsilon}(r-\varepsilon)^{-1}(\sum_{j=0}^{n}(a_\varepsilon(j)-a_r(j))(s_\varepsilon(j+1)s_r(j) + s_\varepsilon(j)s_r(j+1)) \\ &\quad + \sum_{j=1}^{n}(b_\varepsilon(j) - b_r(j))s_\varepsilon(j)s_r(j)) \\ &= \lim_{r\to\varepsilon}\sum_{j=1}^{n}((a_0(j)-a_1(j))(s_\varepsilon(j+1)s_r(j) + s_\varepsilon(j)s_r(j+1)) \\ &\quad + \Delta b(j)s_\varepsilon(j)s_r(j)) \\ &= \sum_{j=1}^{n}(2\Delta a(j)s_\varepsilon(j)s_\varepsilon(j+1) + \Delta b(j)s_\varepsilon(j)^2) \\ &= \sum_{j=1}^{n}(\Delta a(j)s_\varepsilon(j)s_\varepsilon(j+1) + \Delta b(j)s_\varepsilon(j)^2) + \sum_{j=1}^{n+1}\Delta a(j-1)s_\varepsilon(j-1)s_\varepsilon(j) \end{aligned}$$

and hence

$$\begin{aligned} &= \sum_{j=1}^{n} s_\varepsilon(j)(\Delta a(j)s_\varepsilon(j+1) + \Delta a(j-1)s_\varepsilon(j-1) + \Delta b(j)s_\varepsilon(j)) \\ &\quad + \Delta a(n)s_\varepsilon(n)s_\varepsilon(n+1) \\ &= \sum_{j=1}^{n} s_\varepsilon(j)(\Delta\tau s_\varepsilon)(j) + \Delta a(n)s_\varepsilon(n)s_\varepsilon(n+1), \end{aligned}$$

where we used equations (3.11) and (3.4), $s_\varepsilon(n_0) = 0, a_\varepsilon - a_r = (r-\varepsilon)\Delta a$, and $b_\varepsilon - b_r = (r-\varepsilon)\Delta b$. Moreover, we have

$$(\Delta H_{n_0,n+1}\vec{s}_\varepsilon)(j) = \begin{cases} (\Delta\tau s_\varepsilon)(j) & \text{for all } j = 1,\dots,n-1 \\ \Delta a(n-1)s_\varepsilon(n-1) + \Delta b(n)s_\varepsilon(n), \end{cases}$$

and hence

$$\begin{aligned}
&\langle \vec{s}_\varepsilon, \Delta H_{n_0,n+1}\vec{s}_\varepsilon\rangle \\
&\quad = \sum_{j=1}^{n-1} s_\varepsilon(j)(\Delta\tau s_\varepsilon)(j) + s_\varepsilon(n)\Delta a(n-1)s_\varepsilon(n-1) + s_\varepsilon(n)\Delta b(n)s_\varepsilon(n) \\
&\quad = \sum_{j=1}^{n} s_\varepsilon(j)(\Delta\tau s_\varepsilon)(j) - s_\varepsilon(n)\Delta a(n)s_\varepsilon(n+1)
\end{aligned}$$

proves the claim. □

Lemma A.4. *Let $n > n_0 + 2$, then*

$$\dot\theta_\varepsilon(n) = \frac{\langle \vec{s}_\varepsilon, \Delta H_{n_0,n+1}\vec{s}_\varepsilon\rangle}{\rho_\varepsilon(n)^2} = \frac{\langle \vec{s}_\varepsilon, H^0_{n_0,n+1}\vec{s}_\varepsilon\rangle}{\rho_\varepsilon(n)^2} - \frac{\langle \vec{s}_\varepsilon, H^1_{n_0,n+1}\vec{s}_\varepsilon\rangle}{\rho_\varepsilon(n)^2} \tag{A.16}$$

holds for all $\varepsilon \in [0,1]$.

Proof. We have $\frac{d}{d\varepsilon}\frac{1}{a_\varepsilon(n)} = \frac{\Delta a(n)}{a_\varepsilon(n)^2}$, hence

$$\begin{aligned}
\dot s_\varepsilon(n+1) &= \frac{d}{d\varepsilon}(-a_\varepsilon(n)^{-1}\rho_\varepsilon(n)\cos\theta_\varepsilon(n)) \\
&= -\frac{\Delta a(n)}{a_\varepsilon(n)^2}\rho_\varepsilon(n)\cos\theta_\varepsilon(n) - a_\varepsilon(n)^{-1}(\dot\rho_\varepsilon(n)\cos\theta_\varepsilon(n) - \rho_\varepsilon(n)\sin\theta_\varepsilon(n)\dot\theta_\varepsilon(n)) \\
&= a_\varepsilon(n)^{-1}(\Delta a(n)s_\varepsilon(n+1) - \dot\rho_\varepsilon(n)\cos\theta_\varepsilon(n) + s_\varepsilon(n)\dot\theta_\varepsilon(n))
\end{aligned}$$

and

$$\frac{d}{d\varepsilon}(a_\varepsilon(n)s_\varepsilon(n)) = -\Delta a(n)s_\varepsilon(n) + a_\varepsilon(n)(\dot\rho_\varepsilon(n)\sin\theta_\varepsilon(n) + \rho_\varepsilon(n)\cos\theta_\varepsilon(n)\dot\theta_\varepsilon(n)).$$

By

$$\begin{aligned}
&a_\varepsilon(n)s_\varepsilon(n)\dot s_\varepsilon(n+1) - s_\varepsilon(n+1)\frac{d}{d\varepsilon}(a_\varepsilon(n)s_\varepsilon(n)) \\
&\quad = s_\varepsilon(n)(\Delta a_\varepsilon(n)s_\varepsilon(n+1) - \dot\rho_\varepsilon(n)\cos\theta_\varepsilon(n) + s_\varepsilon(n)\dot\theta_\varepsilon(n)) \\
&\qquad + s_\varepsilon(n+1)(\Delta a(n)s_\varepsilon(n) - a_\varepsilon(n)\dot\rho_\varepsilon(n)\sin\theta_\varepsilon(n) \\
&\qquad - a_\varepsilon(n)\rho_\varepsilon(n)\cos\theta_\varepsilon(n)\dot\theta_\varepsilon(n)) \\
&\quad = 2\Delta a(n)s_\varepsilon(n)s_\varepsilon(n+1) - \rho_\varepsilon(n)\sin\theta_\varepsilon(n)\dot\rho_\varepsilon(n)\cos\theta_\varepsilon(n) \\
&\qquad + \dot\rho_\varepsilon(n)\sin\theta_\varepsilon(n)\rho_\varepsilon(n)\cos\theta_\varepsilon(n) + \dot\theta_\varepsilon(n)(s_\varepsilon(n)^2 + a_\varepsilon(n)^2 s_\varepsilon(n+1)^2)
\end{aligned}$$

$$= 2\Delta a(n) s_\varepsilon(n) s_\varepsilon(n+1) + \rho_\varepsilon(n)^2 \dot\theta_\varepsilon(n)$$

and Lemma A.3 the claim holds. □

Lemma A.5. *Let $\Delta J \geqslant 0$, then $\#_{[0,N-1]}(s_0, s_1) \geqslant 0$ and*

$$\#_{[0,N-1]}(u, s_0) \geqslant 2 \implies \#_{[0,N-1]}(u, s_1) \geqslant 1,$$
$$\#_{[0,N-1]}(s_0, u) \geqslant 2 \implies \#_{[0,N-1]}(s_1, u) \geqslant 1.$$

Proof. By Lemma A.4 we have $\dot\theta_\varepsilon(N-1) = \rho_\varepsilon(N-1)^{-2}\langle \vec{s}_\varepsilon, \Delta J \vec{s}_\varepsilon\rangle \geqslant 0$ and hence $\theta_{s_1}(N-1) \geqslant \theta_{s_0}(N-1)$. Thus, by (3.46) we have

$$\begin{aligned}\#_{[0,N-1]}(s_0, s_1) &= \lceil \Delta_{s_0,s_1}(N-1)/\pi\rceil - \lceil \Delta_{s_0,s_1}(0)/\pi\rceil \\ &= \lceil (\theta_{s_1}(N-1) - \theta_{s_0}(N-1))/\pi \rceil \geqslant 0.\end{aligned}$$

Moreover, by Theorem 6.4 we have

$$\#_{[0,N-1]}(u, s_1) \geqslant \#_{[0,N-1]}(u, s_0) + \#_{[0,N-1]}(s_0, s_1) - 1 \geqslant 1$$

and $\#_{[0,N-1]}(s_0, u) \geqslant \#_{[0,N-1]}(s_0, s_1) + \#_{[0,N-1]}(s_1, u) - 1 \geqslant 1$. □

Lemma A.6. *Fix some $n > n_0 + 2$ and let $\varepsilon \in [0,1]$ such that the Weyl m-function*

$$m^{n_0}_{\varepsilon,-}(z, n+1) = \langle \delta_n, (H_{n_0,n+1}(\varepsilon) - z)^{-1}\delta_n\rangle \tag{A.17}$$

exists, that is, let $s_\varepsilon(n+1) \neq 0$, then

$$\begin{aligned}\frac{d}{d\varepsilon} m^{n_0}_{\varepsilon,-}(z, n+1) &= \frac{\langle \vec{s}_\varepsilon, \Delta H_{n_0,n+1}\vec{s}_\varepsilon\rangle}{a_\varepsilon(n)^2 s_\varepsilon(z,n+1)^2} \\ &= \underbrace{\frac{\prod_{j=n_0+1}^{n-1} a(j)^2}{\det(H_{n_0,n+1}(\varepsilon) - z)^2}}_{>0} \langle \vec{s}_\varepsilon, \Delta H_{n_0,n+1}\vec{s}_\varepsilon\rangle.\end{aligned} \tag{A.18}$$

Proof. By (A.9) we have

$$m^{n_0}_{\varepsilon,-}(z, n+1) = \frac{s_\varepsilon(z,n)}{-a_\varepsilon(n) s_\varepsilon(z,n+1)} = \frac{\sin\theta_\varepsilon(n)}{\cos\theta_\varepsilon(n)} = \tan\theta_\varepsilon(n)$$

and hence by Lemma A.4 we have

$$\begin{aligned}\frac{d}{d\varepsilon} m^{n_0}_{\varepsilon,-}(z, n+1) &= \frac{d}{d\varepsilon}\tan\theta_\varepsilon(n) \\ = \frac{\dot\theta_\varepsilon(n)}{\cos^2\theta_\varepsilon(n)} &= \frac{\langle \vec{s}_\varepsilon, \Delta H_{n_0,n+1}\vec{s}_\varepsilon\rangle}{\rho_\varepsilon(n)^2\cos^2\theta_\varepsilon(n)} = \frac{\langle \vec{s}_\varepsilon, \Delta H_{n_0,n+1}\vec{s}_\varepsilon\rangle}{a_\varepsilon(n)^2 s_\varepsilon(z,n+1)^2}.\end{aligned}$$

Moreover by Lemma 5.1 we have $s_\varepsilon(z,n+1) = \frac{\det(H_{n_0,n+1}(\varepsilon)-z)}{\prod_{j=n_0+1}^{n} -a_\varepsilon(j)}$. □

In [46] and [4] a slightly different transformation into Prüfer variables has been used, namely

$$\begin{aligned} u(n) &= \rho_u(n)\sin\theta_u(n), \\ u(n+1) &= \rho_u(n)\cos\theta_u(n). \end{aligned} \tag{A.19}$$

Lemma A.7. *Let $n_0 = 0, n > 2$, and let $\rho_\varepsilon, \theta_\varepsilon$ denote the Prüfer variables from (A.19), then*

$$\dot\theta_\varepsilon(n) = \frac{\langle \vec{s}_\varepsilon, \Delta H_{0,n+1}\vec{s}_\varepsilon\rangle + \Delta a_\varepsilon(n)s_\varepsilon(n)s_\varepsilon(n+1)}{-a_\varepsilon(n)\rho_\varepsilon(n)^2} = \frac{\langle \vec{s}_\varepsilon, \Delta\tau\vec{s}_\varepsilon\rangle}{-a_\varepsilon(n)\rho_\varepsilon(n)^2}. \tag{A.20}$$

Proof. We have $\frac{d}{d\varepsilon}\frac{1}{a_\varepsilon(n)} = \frac{\Delta a(n)}{a_\varepsilon(n)^2}$, hence

$$\dot s_\varepsilon(n+1) = \dot\rho_\varepsilon(n)\cos\theta_\varepsilon(n) - \rho_\varepsilon(n)\sin\theta_\varepsilon(n)\dot\theta_\varepsilon(n)$$

and

$$\begin{aligned} &\frac{d}{d\varepsilon}(a_\varepsilon(n)s_\varepsilon(n)) \\ &\quad = -\Delta a(n)s_\varepsilon(n) + a_\varepsilon(n)(\dot\rho_\varepsilon(n)\sin\theta_\varepsilon(n) + \rho_\varepsilon(n)\cos\theta_\varepsilon(n)\dot\theta_\varepsilon(n)). \end{aligned}$$

By

$$\begin{aligned} &a_\varepsilon(n)s_\varepsilon(n)\dot s_\varepsilon(n+1) - s_\varepsilon(n+1)\frac{d}{d\varepsilon}(a_\varepsilon(n)s_\varepsilon(n)) \\ &\quad = a_\varepsilon(n)s_\varepsilon(n)(\dot\rho_\varepsilon(n)\cos\theta_\varepsilon(n) - s_\varepsilon(n)\dot\theta_\varepsilon(n)) - s_\varepsilon(n+1)(-\Delta a(n)s_\varepsilon(n) \\ &\qquad + a_\varepsilon(n)(\dot\rho_\varepsilon(n)\sin\theta_\varepsilon(n) + \rho_\varepsilon(n)\cos\theta_\varepsilon(n)\dot\theta_\varepsilon(n))) \\ &\quad = a_\varepsilon(n)s_\varepsilon(n)\dot\rho_\varepsilon(n)\cos\theta_\varepsilon(n) - a_\varepsilon(n)s_\varepsilon(n)^2\dot\theta_\varepsilon(n) + s_\varepsilon(n+1)\Delta a(n)s_\varepsilon(n) \\ &\qquad - s_\varepsilon(n+1)a_\varepsilon(n)\dot\rho_\varepsilon(n)\sin\theta_\varepsilon(n) - s_\varepsilon(n+1)a_\varepsilon(n)s_\varepsilon(n+1)\dot\theta_\varepsilon(n) \end{aligned}$$

and thus

$$\begin{aligned} &= s_\varepsilon(n)(a_\varepsilon(n)\dot\rho_\varepsilon(n)\cos\theta_\varepsilon(n) + \Delta a(n)s_\varepsilon(n+1) \\ &\quad - a_\varepsilon(n)\dot\rho_\varepsilon(n)\cos\theta_\varepsilon(n)) - a_\varepsilon(n)\dot\theta_\varepsilon(n)(s_\varepsilon(n)^2 + s_\varepsilon(n+1)^2) \\ &= s_\varepsilon(n)\Delta a(n)s_\varepsilon(n+1) - a_\varepsilon(n)\dot\theta_\varepsilon(n)\rho_\varepsilon(n)^2, \end{aligned}$$

and Lemma A.3 we have

$$\begin{aligned} &\langle \vec{s}_\varepsilon, \Delta H_{0,n+1}\vec{s}_\varepsilon\rangle \\ &\quad + 2\Delta a(n)s_\varepsilon(n)s_\varepsilon(n+1) = s_\varepsilon(n)\Delta a(n)s_\varepsilon(n+1) - a_\varepsilon(n)\dot\theta_\varepsilon(n)\rho_\varepsilon(n)^2 \end{aligned}$$

and hence $\dot\theta_\varepsilon(n) = -(\langle \vec{s}_\varepsilon, \Delta H_{0,n+1}\vec{s}_\varepsilon\rangle + \Delta a(n)s_\varepsilon(n)s_\varepsilon(n+1))a_\varepsilon(n)^{-1}\rho_\varepsilon(n)^{-2}$.

$\square$

Obviously, (A.20) is a generalization of (2.22) in [46], where $\Delta J = \mathbb{I}$ and

$$\dot{\theta}_\varepsilon(n) = \frac{\sum_{j=1}^{n} s_\varepsilon(j)^2}{-a(n)\rho_\varepsilon(n)^2} \tag{A.21}$$

holds and further, (A.20) agrees with (3.3) in [4], where we have $\Delta a = 0$ and

$$\dot{\theta}_\varepsilon(n) = \frac{\sum_{j=1}^{n} \Delta b(j) s_\varepsilon(j)^2}{-a(n)\rho_\varepsilon(n)^2}. \tag{A.22}$$

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